

The University of Chicago

THE HAMILTON-JACOBI THEORY
FOR THE PROBLEMS OF
BOLZA AND MAYER

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1939

By

MARY KENNY LANDERS

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1. Introduction. The problem of Bolza in non-parametric form is that of finding in a class of arcs

$$(1:1) \quad y_1(t) \quad (i = 1, \dots, n; t_1 \leq t \leq t_2)$$

satisfying the differential equations and end-conditions of the form

$$\phi_{\alpha}(t, y, y') = 0 \quad (\alpha = 0, 1, \dots, m < n-1),$$

$$\psi_{\mu}[t_1, y(t_1), t_2, y(t_2)] = 0 \quad (\mu = 0, 1, \dots, p \leq 2n+1),$$

one which minimizes a sum of the form

$$J = g[t_1, y(t_1), t_2, y(t_2)] + \int_{t_1}^{t_2} f(t, y, y') dt.$$

The symbols y and y' represent the set $(y_1, y_2, \dots, y_n)$ and its derivative $(y_1', y_2', \dots, y_n')$.

In parametric form the problem is that of finding in a class of arcs

$$y_1(t) \quad (i = 1, \dots, n; t_1 \leq t \leq t_2)$$

satisfying the differential equations and end-conditions of the form

$$(1:2) \quad \phi_{\beta}(y, y') = 0 \quad (\beta = 1, \dots, m < n-1),$$

$$(1:3) \quad \psi_{\sigma}[y(t_1), y(t_2)] = 0 \quad (\sigma = 1, \dots, p \leq 2n),$$

one which minimizes a function of the form

$$(1:4) \quad J = g[y(t_1), y(t_2)] + \int_{t_1}^{t_2} f(y, y') dt.$$

The functions ϕ_{σ} and the function f are assumed to be positively homogeneous of order one in the derivatives y' in order that the solutions of equations (1:2) and the integral in (1:4), will be independent of the parameter t .

The parametric problem in y -space may be considered as a non-parametric problem in ty -space if, to equations (1:2) and (1:3), there are adjoined, respectively, the equations

$$(1:5) \quad \phi_0(y, y') = \phi(y, y') - 1 = 0,$$

$$(1:6) \quad \psi_0 = t_1 = 0,$$

which define the parameter t uniquely. The function ϕ is a conveniently chosen function of the variables y and y' , positively homogeneous of order one in y' .

These three formulations of the problem of Bolza will be referred to as problems A, B, C, respectively. The theory of problem B will be deduced from that of problem C, which is a special case of problem A.

The problem of Mayer with variable end-points in non-parametric form as formulated by Bliss is the problem A with its integrand function f identically zero. It reduces to the classical problem of Mayer when the expression to be minimized is the function $g = y_1(t_2)$ and the end-conditions $\psi_{\mu} = 0$ are $t_1 - \alpha_1 = y_{11} - \beta_{11} = t_2 - \alpha_2 = y_{j2} - \beta_{j2} = 0$, ($i = 1, \dots, n$; $j = 2, \dots, n$) the α 's and β 's being constants. Similarly, the problem of Mayer with

variable end-points in parametric form is the problem B with its integrand function f identically zero. It reduces to the classical problem of Mayer when the expression to be minimized is $g = y_1(t_2)$ and the end-conditions $\psi_c = 0$ are $y_1(t_1) - \beta_{11} = y_j(t_2) - \beta_{j2} = 0$.

It is the purpose of this paper to extend the known Hamilton-Jacobi and field theory for the problem of Lagrange in parametric and non-parametric form to the problems of Bolza and Mayer in parametric and non-parametric form. In general, the theory for the parametric problem is deduced from that of the non-parametric problem by a method analogous to that used by Teach (X)*. Throughout the paper the emphasis is placed upon the Hamilton function H and, in particular, the field theory is based upon the definition of a field in terms of the canonical variables given in Section 6. The corresponding theory in terms of the function F defined in Section 3 is readily obtained with the aid of the definition of the function H given in Section 4.

The analytic basis of the problem is given in Section 2. The multiplier rule and its usual corollaries are given in Section 3 for both problems A and B. It is shown that along a minimizing arc for problem C the multiplier λ_0 associated with ϕ_0 has the value zero. This result agrees with that obtained by Teach for the problem of Lagrange and was deduced by the same method.

In Section 4 the imbedding theorem for the extremals of problem B is deduced from the corresponding theorem for problem A. The Hamilton function H is defined, and some of its properties are determined. The homogeneity of the function H for the problem of

* Roman numerals in parentheses refer to the bibliography at the end of the paper.

Mayer in parametric form and its effect upon the imbedding theorem are deduced from the theory developed for the non-parametric problem by Bliss and Hestenes (III). The necessary conditions of Weierstrass and Clebsch are given in regular and canonical variables in Section 5.

In addition to the definition of a Mayer field for problem A in terms of regular variables due to Bliss, a second definition of a Mayer field in terms of the canonical variables and the Hamilton function H is given in Section 6. The field theory developed in this paper is based upon the second definition.

The next two sections are devoted to a discussion of transversal surfaces of a Mayer field for problem A, the construction of fields from a $2n$ -parameter family of extremals, and the relations between the extremals and a complete integral of the Hamilton-Jacobi equation. In Section 8 a necessary and sufficient condition for the non-appearance of some of the parameters in the functions defining the extremal arcs of a field is given in terms of the function defining the transversal surfaces of the field. This result enables us to obtain at once from the Hamilton-Jacobi theory for the problem of Bolza further results for the problem of Mayer and other problems whose extremals are abnormal on every subinterval.

Fields for the problem of Mayer have some unusual properties on account of the homogeneity of the Hamilton function H . The form which they take for the non-parametric problem is discussed in Section 9. In particular, it is shown that the extremals of a field lie on its transversal surfaces and that each transversal surface satisfies the definition of a field given by Kneser (IX, 43) for the problem of Mayer. A method is given for constructing an

n -parameter family of Mayer fields from a $2n$ -parameter family of extremals. It is also shown how to determine a $2n$ -parameter family of extremals from a complete integral of the Hamilton-Jacobi equation. Finally, the theorem deduced in the previous section for problem A concerning the non-appearance of some of the parameters in the functions defining the extremal arcs of a field is applied to this problem.

Two definitions of a Mayer field for problem B are given in Section 10 which are analogous to those given in Section 6 for problem A. The field theory for problem B is based upon the definition of a field in terms of the canonical variables and the Hamilton function H . This section also includes the definition of the transversal surfaces of a field and a discussion of their properties.

The form which the field theory for problem A assumes for problem C is given in Section 11. The only fields for problem C which are important for problem B are those which are independent of the parameter t . It is shown how to construct such fields from a $2n$ -parameter family of extremals, using a method suggested by the proof of a similar theorem by Bliss for a parametric problem (V, 241). In Sections 12 and 13 the theory for problem B and the problem of Mayer in parametric form analogous to that developed in Sections 8 and 9 for the corresponding non-parametric problems is deduced from the theory developed in Section 11 for problem C.

The relation between the foregoing theory for the problem of Mayer in parametric and non-parametric form and the theory of Kneser (IX) is discussed in Sections 14 and 15. In particular, it is shown that a Mayer field as defined by Bliss consists of a one-parameter family of fields as defined by Kneser, and that every

field of the latter type can be imbedded in a one-parameter family of such fields which form a field as defined by Bliss.

2. Analytic basis for the problem. The following definitions and assumptions will be made in order to carry through the analysis that follows:

For the problem A it is assumed that the functions ϕ_α and f have continuous partial derivatives of at least the first three orders on a $(2n+1)$ -dimensional region D_1 of points (t, y, y') and that the matrix $||\phi_\alpha y'_1||$ has rank $m+1$ on this domain. The functions g, ψ_μ are assumed to have continuous second derivatives on a region D_2 of $(2n+2)$ -dimensional points $[t_1, y(t_1), t_2, y(t_2)]$ and to be such that on this domain the matrix

$$||\psi_\mu t_1 \quad \psi_\mu y_{11} \quad \psi_\mu t_2 \quad \psi_\mu y_{12}||$$

has rank $p+1$. An arc in D_1 is an arc of type $(1:1)$ which is continuous and consists of a finite number of sub-arcs on each of which the functions $y_1(t)$ have continuous derivatives and whose elements (t, y, y') are all interior to D_1 . An admissible arc is an arc in D_1 with its end-element $[t_1, y(t_1), t_2, y(t_2)]$ interior to D_2 . On every such arc the functions ϕ_α, ψ_μ, g and J have well-defined values.

For the problem B it is assumed that the functions ϕ_β and f have continuous partial derivatives of at least the first three orders and are positively homogeneous of order one in y' on a $2n$ -dimensional region R_1 of points (y, y') which is such that if (y, y') is in R_1 , then (y, ky') for k positive is also in R_1 , and $y'_1 y'_1 \neq 0$. The matrix $||\phi_\beta y'_1||$ is assumed to have rank m on this domain. The functions g, ψ_σ are assumed to have continuous second derivatives on a region R_2 of $2n$ -dimensional

points $[y_{11}, y_{12}]$ and to be such that on this domain the matrix $||\Psi_{\sigma y_{11}} \quad \Psi_{\sigma y_{12}}||$ has rank p . The definitions of an arc in R_1 and an admissible arc are analogous to those made for problem A.

For the problem C, which is a special form of problem A obtained from problem B as described above, the region $\mathfrak{D}_1$ corresponding to D_1 is taken to be the set of all elements (t, y, y') such that (y, y') is in R_1 . The function $\phi(y, y')$, appearing in ϕ_0 , is chosen to be positive and to be positively homogeneous of order one in y' on $\mathfrak{D}_1$. Moreover, this function is selected so as to have continuous third derivatives and to be such that the extended matrix $||\phi_{\alpha y'_i}||$, ($\alpha = 0, 1, \dots, m$), has rank $m+1$ on $\mathfrak{D}_1$. For example, the function $\phi = (y'_1 y'_1)^{1/2}$ has these properties. The region $\mathfrak{D}_2$ is taken to be the set of all points $[t_1, y(t_1), t_2, y(t_2)]$ such that $[y(t_1), y(t_2)]$ is in R_2 .

Every admissible arc for problem B, satisfying the equations $\phi_\beta(y, y') = 0$ and the end-conditions $\Psi_\sigma = 0$, can be changed uniquely by altering the parameter into one for problem C with the same properties, with $t_1 = 0$, and satisfying the equation $\phi_0 = \phi(y, y') - 1 = 0$. Hence, problem B is equivalent to problem C in the sense that there exists a one-to-one correspondence between their admissible arcs satisfying the differential equations and end-conditions such that corresponding arcs give to J the same value. It should be observed, however, that if E and E' represent corresponding arcs for problems B and C, respectively, then not every arc K in a neighborhood of E can be transformed into an arc K' lying in a prescribed neighborhood of E' . This is readily seen if the parameter t represents arc length since then the arc K can be chosen so that the length of K' exceeds that of E' by an arbitrary amount. The theory deduced for problem B is independent

of the particular parameterization of the arcs used, but in this paper whenever the two problems B and C are being compared it will be understood that corresponding arcs have the parameterization determined by problem C.

A set $\xi_1, \xi_2, \eta_1(t)$, where ξ_1, ξ_2 represent constants, is called a set of admissible variations along an admissible arc E for problem A provided that $\eta_1(t)$ possesses the continuity properties of an admissible arc. The equations of variation of the functions ϕ_α, ψ_μ along E are given by the equations

$$(2:1) \quad \bar{I}_\alpha = \phi_\alpha y_1 \eta_1 + \phi_\alpha y_1' \eta_1' = 0 \quad (\alpha = 0, 1, \dots, m),$$

$$(2:2) \quad \bar{\Psi}_\mu [\xi_1, \eta_1(t_1), \xi_2, \eta_1(t_2)] = 0 \quad (\mu = 0, 1, \dots, p),$$

where

$$\begin{aligned} \bar{\Psi}_\mu = & (\psi_{\mu t_1} + y_{11}' \psi_{\mu y_{11}}) \xi_1 + \psi_{\mu y_{11}} \eta_1(t_1) \\ & + (\psi_{\mu t_2} + y_{12}' \psi_{\mu y_{12}}) \xi_2 + \psi_{\mu y_{12}} \eta_1(t_2). \end{aligned}$$

For problem C, $\bar{\Psi}_0 = \xi_1 = 0$.

Similar definitions hold for problem B except that constants ξ_1, ξ_2 are not present. Equations (2:1) and (2:2) take the forms

$$(2:3) \quad \bar{I}_\beta = \phi_\beta y_1 \eta_1 + \phi_\beta y_1' \eta_1' = 0 \quad (\beta = 1, \dots, m),$$

$$(2:4) \quad \bar{\Psi}_\sigma = \psi_{\sigma y_{11}} \eta_1(t_1) + \psi_{\sigma y_{12}} \eta_1(t_2) = 0 \quad (\sigma = 1, \dots, p).$$

The functions $\eta_1(t)$ for problems B and C are understood to have the same parameterization. It should be noted that if E is an admissible arc for problem B defined by functions $y_1(t)$ having continuous second derivatives for $t_1 \leq t \leq t_2$ and if the functions $\eta_1(t)$ form a set of admissible variations satisfying the equations

(2:3) along E , then, on account of the homogeneity of the functions ϕ_β , the functions $\zeta_1(t) = \eta_1 + \xi(t)y_1'(t)$, where $\xi(t)$ possesses the continuity properties of $\eta_1(t)$, are also solutions of equations (2:3). This property will be used in the proof of the following lemma.

LEMMA 2:1. Let E be an admissible arc for problem B defined by functions $y_1(t)$ having continuous second derivatives on the interval $t_1 \leq t \leq t_2$. If $\eta_1(t)$ is a set of admissible variations satisfying the equations (2:3) along E , then ξ_1, ξ_2 ,

$\zeta_1(t) = \eta_1(t) - \xi(t)y_1'(t)$, where

$$\xi(t) = \int_{t_1}^t (\phi_{y_1} \eta_1 + \phi_{y_1'} \eta_1') dt + c, \quad \xi_1 = \xi(t_1), \quad \xi_2 = \xi(t_2)$$

forms a set of admissible variations satisfying the equations (2:1) along the corresponding arc E' for problem C. Conversely, if $\xi_1, \xi_2, \zeta_1(t)$ is a set of admissible variations satisfying equations (2:1) along E' , then the set $\eta_1(t) = \zeta_1(t) + \xi(t)y_1'$ is a set of admissible variations satisfying the equations (2:3) along E , where $\xi(t)$ is an arbitrary function having $\xi(t_1) = \xi_1, \xi(t_2) = \xi_2$ and continuity properties like those of $\zeta_1(t)$. In either case $\eta_1(t)$ satisfy equations (2:4) if and only if the set $\xi_1, \xi_2, \zeta_1(t)$ satisfy the equations (2:2) for problem C.

This result can be established by substitution with the help of the remark preceding this lemma.

An admissible arc E for problem A satisfying the equations $\phi_\alpha = 0$ is said to be normal relative to the end-conditions, and designated simply as normal, if there exist for it $p+1$ sets of admissible variations $\xi_{1v}, \xi_{2v}, \eta_{1v}(t)$ satisfying equations (2:1) such that the determinant $|I_\mu(\xi_v, \eta_v)|$ is different from zero for $(\mu, v = 0, 1, \dots, p)$. Similarly, an admissible

arc E for problem B satisfying the equations $\delta \beta = 0$ is said to be normal if there exist for it p sets of admissible variations $\gamma_{1q}(t)$ satisfying equations (2:3) such that the determinant $|\Psi_{\sigma}(\gamma_q)|$ is different from zero for $(\sigma, q = 1, \dots, p)$.

LEMMA 2:2. Let E be an admissible arc for problem B defined by functions $y_1(t)$ having continuous second derivatives on the interval $t_1 \leq t \leq t_2$. Then E is normal if and only if its corresponding arc for problem C is normal.

Corresponding arcs for problems B and C have the same continuity properties since, as indicated previously, when the arcs are being compared they are assumed to have the same parameterization. If an admissible arc E' for problem C having the continuity properties stated in the lemma for the arc E is normal, a determinant of the form

$$\left| \Psi_{\sigma y_{11}} [\zeta_{1v}(t_1) + \xi_{1v} y_1'(t_1)]^{\xi_{1v}} + \Psi_{\sigma y_{12}} [\zeta_{1v}(t_2) + \xi_{2v} y_1'(t_2)] \right| \quad (2:5)$$

$(v = 0, 1, \dots, p; \sigma = 1, \dots, p)$

must be different from zero. It may be assumed without loss of generality that $\xi_{10} = 1$, $\xi_{1v} = 0$ ($v \neq 0$). Then the determinant (2:5) is equal to the determinant

$$\left| \Psi_{\sigma y_{11}} \zeta_{1q}(t_1) + \Psi_{\sigma y_{12}} [\zeta_{1q}(t_2) + \xi_{2q} y_1'(t_2)] \right| \quad (\sigma, q = 1, \dots, p).$$

But the sets $\gamma_{1q} = \zeta_{1q}(t) + \xi_q(t) y_1'$, described in the second part of Lemma 2:1, are p sets of admissible variations satisfying equations (2:3) along the corresponding arc E for problem B. The arc E, consequently, is normal. Conversely, if the determinant $|\Psi_{\sigma}(\gamma_{1q})|$ is different from zero for p sets of admissible variations $\gamma_{1q}(t)$ satisfying the equations (2:3) along an arc E for

problem B, then $\xi_{10} = 1$, $\xi_{20} = 0$, $\xi_{10} = 0$ and ξ_{1q} , ξ_{2q} , ξ_{1q} , where the latter sets are related to η_{1q} as described in the first part of Lemma 2:1, are $p+1$ sets of admissible variations satisfying equations (2:1) along the corresponding arc E' for problem C for which the determinant (2:5) is different from zero. The arc E' , therefore, is normal.

It will be assumed throughout this paper that the minimizing arc for the problem under discussion is normal.

3. The multiplier rule. From the theory of the problem of Bolza it is evident at once that a normal minimizing arc for problem A must satisfy the following multiplier rule with a unique set of multipliers (IV, 26).

THEOREM 3:1. If E is a normal minimizing arc for problem A, there exist a unique set of constants c , c_1 , e_μ and a unique function

$$(3:1) \quad F(t, y, y', \lambda) = f + \lambda_\alpha(t) \phi_\alpha \quad (\alpha = 0, 1, \dots, m)$$

such that the equations

$$(3:2) \quad F - y_1' F_{y_1'} = \int_{t_1}^t F_t dt + c, \quad F_{y_1'} = \int_{t_1}^t F_{y_1} dt + c_1, \quad \phi_\alpha(t, y, y') = 0$$

hold along E, and furthermore such that the transversality condition

$$(3:3) \quad [(F - y_1' F_{y_1'}) dt + F_{y_1'} dy_1]_1^2 + dg + e_\mu d\psi_\mu = 0$$

holds at the ends of E for every choice of the differentials dt_1 , dt_2 , dy_{11} , dy_{12} . Along every subarc of E between corners the multipliers $\lambda_\alpha(t)$ are continuous. If $f \equiv 0$, the multipliers $\lambda_\alpha(t)$ are either identically zero or else do not vanish simultaneously at any point.

The functions $F - y_1^1 F_{y_1^1}$ and $F_{y_1^1}$ are continuous along a minimizing arc E . On each subarc of E between corners they have continuous derivatives

$$(3:4) \quad d(F - y_1^1 F_{y_1^1})/dt = F_t, \quad dF_{y_1^1}/dt = F_{y_1^1}.$$

The equations

$$(3:5) \quad dF_{y_1^1}/dt = F_{y_1^1}, \quad 0 = \phi_\alpha(t, y, y') \quad (1 = 1, \dots, n; \alpha = 0, 1, \dots, m)$$

are called the Euler-Lagrange equations. If the arc E consists of a single arc without corners and with continuous derivatives $y_1''(t)$, $\lambda_\alpha'(t)$, then the first equation in (3:4) is a consequence of equations (3:5) (IV, 32). If the matrix

$$(3:6) \quad R = \begin{vmatrix} F_{y_1^1 y_k^1} & \phi_\gamma y_1^1 \\ \phi_\alpha y_k^1 & 0 \end{vmatrix} \quad \begin{matrix} (1, k = 1, \dots, n) \\ (\alpha, \gamma = 0, 1, \dots, m) \end{matrix}$$

has rank $n+m+1$ at a point of a subarc of E between corners, then the functions $y_1^1(t)$, $\lambda_\alpha(t)$ belonging to the arc have continuous derivatives of at least the first order with respect to t at that point.

THEOREM 3:2. For problem C, the equation

$$(3:7) \quad \lambda_0 = y_1^1 \bar{F}_{y_1^1} - \bar{F}$$

holds identically where $\bar{F}$ is the function for problem C corresponding to the function F for problem A and λ_0 is the multiplier of $\phi_0 = \phi - 1$ in $\bar{F}$. Along a minimizing arc E' , the multiplier λ_0 is equal to zero.

The function $\bar{F}$ is given by the equation

$$\bar{F}(y, y', \lambda) = \lambda_0(\phi - 1) + F,$$

$$F(y, y', \lambda) = f(y, y') + \lambda_\beta(t) \phi_\beta(y, y') \quad (\beta = 1, \dots, m).$$

From the homogeneity relations $F(y, ky', \lambda) = kF(y, y', \lambda)$, $\phi(y, ky') = k\phi(y, y')$, where $k > 0$, it is found by differentiating with respect to k and setting $k = 1$ that

$$(3:8) \quad F = y_1' \bar{F}_{y_1'}, \quad \phi = y_1' \phi_{y_1'},$$

and hence that $\bar{F} = \lambda_0(\phi - 1) + F = y_1' \bar{F}_{y_1'} - \lambda_0$. This proves the identity (3:7). Since $\bar{F}_t = 0$, the multiplier λ_0 has a constant value along a minimizing arc E' satisfying the equations $\phi_\alpha = 0$ and the first equation in (3:2). The transversality condition for E' requires that the coefficient of dt in (3:3), namely $-\lambda_0(t_2)$, must be zero. Hence, along E' the multiplier λ_0 is identically zero.

The function $\bar{F}$ along a minimizing arc E' for problem C is independent of the choice of ϕ and is equal to F . It is now possible to state the following multiplier rule for the corresponding arc E for problem B.

THEOREM 3:3. If E is a normal minimizing arc for problem B, there exist unique constants c_1 , e_μ and a unique function

$$F(y, y', \lambda) = f(y, y') + \lambda_\beta(t) \phi_\beta(y, y') \quad (\beta = 1, \dots, m)$$

such that the equations

$$F_{y_1'} = \int_{t_1}^t F_{y_1} dt + c_1, \quad \phi_\beta(y, y') = 0$$

hold along E , and furthermore such that the transversality condition

$$F_{y_1'} dy_1 \Big|_1^x + dg + e_\mu d\psi_\mu = 0$$

holds at the ends of E for every choice of the differentials dy_{11}, dy_{12} . Along every subarc of E between corners the multipliers $\lambda_\beta(t)$ are continuous and do not vanish simultaneously at any point.

This result follows from Theorem 3:1, Theorem 3:2 and Lemma 2:2 only if the functions $y_1^i(t), \lambda_\beta(t)$ belonging to the minimizing arc have continuous first derivatives. The theorem is, however, true as stated, but we shall not pause to establish it at this time.

The functions $F_{y_1^i}$ are continuous along a minimizing arc E. On each subarc of E between corners they have continuous derivatives $dF_{y_1^i}/dt = F_{y_1^i}$. The equations

$$(3:9) \quad dF_{y_1^i}/dt = F_{y_1^i}, \quad 0 = \phi_\beta(y, y')$$

are called the Euler-Lagrange equations for problem B. The matrix for problem B, corresponding to the matrix R in (3:6) for problem A, is

$$(3:10) \quad R = \left\| \begin{array}{cc} F_{y_1^i y_k^i} & \phi_{\nu y_k^i} \\ \phi_{\beta y_1^i} & 0 \end{array} \right\| \quad \begin{array}{l} (i, k = 1, \dots, n) \\ (\beta, \nu = 1, \dots, m). \end{array}$$

Its rank is less than $n+m$ along an arc E satisfying the equations $\phi_\beta = 0$ because of the homogeneity of the functions F and ϕ_β and the identities

$$(3:11) \quad y_1^i F_{y_1^i y_k^i} = 0$$

which are obtained by differentiating each of the first n identities in (3:8) with respect to y_k^i . Along an arc E' for problem C for which $\lambda_0 = 0, \phi_\alpha = 0$, the matrix R in (3:6) assumes the form

$$(3:12) \quad D = \left\| \begin{array}{cc} F_{y_1' y_k'} & \phi_{\gamma y_k'} \\ \phi_{\alpha y_1'} & 0 \end{array} \right\| \quad \begin{array}{l} (1, k = 1, \dots, n) \\ (\alpha, \gamma = 0, 1, \dots, m). \end{array}$$

THEOREM 3:4. Let E be an arc for problem B satisfying the equations $\phi_{\beta} = 0$ which corresponds to an arc E' for problem C satisfying the equations $\phi_{\alpha} = 0$ and having the multiplier λ_0 equal to zero. Then if the matrix R in (3:10) is of rank $n+m-1$ along E, the matrix D in (3:12) is of rank $n+m+1$ along E', and conversely.

If D has rank less than $n+m+1$, there exist constants λ_1, μ_{α} not all zero satisfying the equations

$$\begin{aligned} \lambda_1 F_{y_1' y_k'} + \mu_{\gamma} \phi_{\gamma y_k'} &= 0 \\ \lambda_1 \phi_{\alpha y_1'} &= 0. \end{aligned}$$

If each k th one of the first n equations is multiplied by y_k' and the n equations are added, the equation $\mu_0 = 0$ is obtained because of the conditions (3:8) and (3:11). It follows that the matrix

$$\left\| \begin{array}{cc} F_{y_1' y_k'} & \phi_{\gamma y_k'} \\ \phi_{\alpha y_1'} & 0 \end{array} \right\| \quad \begin{array}{l} (1, k = 1, \dots, n) \\ (\gamma = 1, \dots, m) \\ (\alpha = 0, 1, \dots, m) \end{array}$$

must have rank less than $n+m$. Applying this same process to the columns of any matrix obtained from this one by deleting one of the first n or one of the last m rows, it is found that the rank of the matrix R must be less than $n+m-1$. On the other hand, if the rank of R is less than $n+m-1$, by adjoining one row and one column to obtain D, it is possible to increase the rank by at most two. Hence, the rank of D would have to be less than $n+m+1$.

THEOREM 3:5. The differentiability condition. On each subarc of a normal minimizing arc E for problem B on which the functions $y_1(t)$ defining E have continuous derivatives and the matrix R in (3:10) has rank $n+m-1$, the functions $y_1'(t)$, $\lambda_\beta(t)$ belonging to E have continuous derivatives of at least the first order with respect to t .

This result follows from the differentiability condition for problem C by means of Theorem 3:4 and the last statement in Theorem 3:2. These theorems are applicable here since this result is independent of normality.

4. The extremals and the canonical variables. An extremal for problem A is defined to be an arc without corners in D_1 and a set of multipliers

$$(4:1) \quad y_1(t), \lambda_\alpha(t) \quad (1 = 1, \dots, n; \alpha = 0, 1, \dots, m; t_1 \leq t \leq t_2)$$

for which the functions $y_1'(t)$, $\lambda_\alpha(t)$ have continuous partial derivatives and satisfy the Euler-Lagrange equations (3:5). For an extremal, the first equation in (3:4) is a consequence of the equations (3:5) (IV, 32). An extremal is said to be non-singular if along it the matrix R in (3:6) has rank $n+m+1$. An extremal for problem B is defined to be an arc without corners in R_1 and a unique set of multipliers

$$(4:2) \quad y_1(t), \lambda_\beta(t) \quad (1 = 1, \dots, n; \beta = 1, \dots, m; t_1 \leq t \leq t_2)$$

for which the functions $y_1'(t)$, $\lambda_\beta(t)$ have continuous derivatives and satisfy the Euler-Lagrange equations (3:9). It is said to be non-singular if along it the matrix R in (3:10) has rank $n+m-1$.

In Section 8 of his lectures on the problem of Bolza (IV), Bliss gives a method of imbedding a non-singular extremal arc E for problem A in a $2n$ -parameter family of such extremals by introducing the canonical variables (t, y, z) related to the variables (t, y, y', χ) by the equations

$$(4:3) \quad z_1 = F_{y_1'}(t, y, y', \chi), \quad 0 = \phi_\alpha(t, y, y') \quad (i=1, \dots, n; \alpha=0, 1, \dots, m).$$

Along E these equations define functions $z_1(t)$ with continuous derivatives on t_1, t_2 and the functional determinant of the right members with respect to y', χ is the determinant of the matrix R in (3:6). By the usual implicit function theorems there exists a neighborhood N of the solutions (t, y, z) belonging to E in which the equations (4:3) have solutions

$$(4:4) \quad y_1' = P_1(t, y, z), \quad \chi_\alpha = L_\alpha(t, y, z) \quad (i=1, \dots, n; \alpha=0, 1, \dots, m)$$

reducing to $y_1'(t), \chi_\alpha(t)$ at each point $[t, y(t), z(t)]$ belonging to E , and which are such that P_1, L_α have continuous derivatives of at least the second order. In the neighborhood N every solution of the equations

$$(4:5) \quad y_1' = P_1(t, y, z), \quad z_1' = F_{y_1'}[t, y, P(t, y, z), L(t, y, z)]$$

defines by means of equations (4:4) functions $L_\alpha(t)$ with which it satisfies equations (4:3) and hence the equations (3:5).

In the neighborhood N the Hamilton function $H(t, y, z)$ may be defined by the equation

$$(4:6) \quad H(t, y, z) = z_1 P_1(t, y, z) - F(t, y, P, L)$$

which is the function into which the expression $y_1' F_{y_1'} - F$ is transformed by the substitution of the values of y_1' and χ_α from equations

(4:4). From equations (4:3) and simple calculations it follows that the partial derivatives of the function H have the values

$$(4:7) \quad H_t = -F_t, \quad H_{y_1} = -F_{y_1}, \quad H_{z_1} = P_1.$$

Hence, equations (4:5) are equivalent to the system

$$(4:8) \quad y_1' = H_{z_1}, \quad z_1' = -H_{y_1}.$$

These are called the canonical equations of the extremals. Some of the consequences of the above theory are summarized in the following theorem, proved by Bliss (IV, 36).

THEOREM 4:1. The imbedding theorem for problem A. Every non-singular extremal E for problem A is embedded for values $t_1 \leq t \leq t_2$, t_0 , a_{10} , ..., a_{n0} , b_{10} , ..., b_{n0} in a $2n$ -parameter family of extremals defined by functions of the form

$$(4:9) \quad y_1(t, t_0, a, b), \quad z_1(t, t_0, a, b)$$

the functions y_1 , y_{1t} , z_1 , z_{1t} have continuous partial derivatives of at least the second order in a neighborhood of the values (t, a, b) defining E, and the matrix

$$\left\| \begin{array}{cc} y_{1a_k} & y_{1b_k} \\ z_{1a_k} & z_{1b_k} \end{array} \right\| \quad (1, k = 1, \dots, n)$$

has rank $2n$ along E.

With the aid of equations (4:8), the following property of the function H is readily established.

THEOREM 4:2. Along every solution $y_1(t)$, $z_1(t)$ of equations (4:8),

$$(4:10) \quad dH/dt = H_t.$$

For, if each 1-th one of the first n equations in (4:8) is multiplied by z_1' and the second n by $-y_1'$ and if then the entire set is summed, the equation $0 = H_{y_1} y_1' + H_{z_1} z_1'$ is obtained, from which the equation (4:10) follows.

THEOREM 4:3. Along every solution $y_1(t)$, $z_1(t)$ of equations (4:8) for problem C, the equation

$$(4:11) \quad H(y, z) = L_0(y, z)$$

holds identically; furthermore, $H(y, z)$ has a constant value not identically zero.

The relation (4:11) follows from condition (3:7). The last part of the theorem follows from the first equation in (3:4), as well as from Theorem 4:2, and the hypothesis that $y_1' y_1' \neq 0$.

THEOREM 4:4. In the functions (4:9) for problem C, one of the parameters b can be taken equal to H .

By Theorems 4:3 and 4:1, the equation $\chi_0 = H(a, b)$ is satisfied by the values (a, b, χ_0) belonging to E . Since not all of the functions H_{z_1} vanish at any point of E , there is a neighborhood of the values (a, b) defining E in which one of them, say H_{z_n} , remains different from zero. By implicit function theory the equation $\chi_0 = H(a, b)$ may be solved for b_n in the neighborhood just described, and

$$(4:12) \quad b_n = b_n(a_1, \dots, a_n, b_1, \dots, b_{n-1}, \chi_0).$$

THEOREM 4:5. The extremals for problem B satisfy the equations

$$y_1' = H_{z_1}(y, z), \quad z_1' = -H_{y_1}(y, z), \quad H(y, z) = 0.$$

This theorem follows at once from Theorem 4:3 since the extremals for problem B are the projections in the y -space of those extremals for problem C for which $L_0(y, z)$ is equal to zero.

THEOREM 4:6. The imbedding theorem for problem B. Every non-singular extremal E for problem B in y -space is imbedded for values $t_{10} \leq t \leq t_{20}$, A_{10} , ..., $A_{n-1,0}$, B_{10} , ..., $B_{n-1,0}$ in a $(2n-2)$ -parameter family of extremals defined by functions of the form

$$Y_1(t, A, B), \quad Z_1(t, A, B).$$

The functions Y_1 , Y_{1t} , Z_1 , Z_{1t} have continuous partial derivatives of at least the second order in a neighborhood of the values (t, A, B) defining E and the matrix

$$(4:13) \quad \left\| \begin{array}{ccc} Y_1' & Y_{1A_k} & Y_{1B_k} \\ Z_1' & Z_{1A_k} & Z_{1B_k} \end{array} \right\| \quad (i = 1, \dots, n; k = 1, \dots, n-1)$$

has rank $2n-1$ along E.

By Theorem 4:4, the functions (4:9) for problem C may be written in the form

$$(4:14) \quad \begin{aligned} y_1[t, a, B, b_n(a, B, H)] &= Y_1(t, a, B, H) \\ z_1[t, a, B, b_n(a, B, H)] &= Z_1(t, a, B, H), \end{aligned}$$

in which the symbols a, B represent the set $(a_1, \dots, a_n, b_1, \dots, b_{n-1})$. The projections in the y -space of the $2n$ -parameter family of solutions of equations (4:8) for problem C, for which $L_0(y, z) = 0$, form a $(2n-2)$ -parameter family of curves in the y -space. For, if an extremal projection in the y -space intersects a sub-space $a_1 = \xi_1(A_1, \dots, A_{n-1})$, having continuous second partial derivatives

in a neighborhood of the intersection and is not tangent to it, then every other projection near the extremal will also intersect without being tangent to it. Hence, the point of intersection of each extremal with the sub-space can be taken as the initial point ξ_1 of the extremal for $t = t_{10}$. Since the only projections important for problem B are those for which $\lambda_0 = 0$ and since $\lambda_0 = L_0(y, z) = H(y, z)$ is a constant not identically zero along every extremal for problem C, the number of essential parameters is further reduced by one. The extremals for problem B are then defined by functions of the form

$$(4:15) \quad Y_1(t, A, B), \quad Z_1(t, A, B)$$

obtained from (4:14) by replacing a_1 by $\xi_1(A_1, \dots, A_{n-1})$ and setting $H = 0$.

Because of the identities

$$a_1 = \xi_1(A) = Y_1(t_{10}, A, B), \quad B_j = Z_j(t_{10}, A, B),$$

$$b_n[\xi(A), B] = Z_n(t_{10}, A, B)$$

and the following relations deduced from them by differentiation with respect to the variables A_j and B_j ,

$$\xi_{1A_j} = Y_{1A_j}, \quad 0 = Y_{1B_j}, \quad 0 = Z_{rA_j}, \quad \delta_{rj} = Z_{rB_j},$$

$$b_{na_1} \xi_{1A_j} = Z_{nA_j}, \quad b_{nB_j} = Z_{nB_j},$$

it is seen that at $t = t_{10}$ the matrix (4:13) is equal to the matrix

$$\left\| \begin{array}{ccc} Y_1' & \xi_{1A_k} & 0 \\ Z_j' & 0 & \delta_{rk} \\ Z_n' & b_{na_1} \xi_{1A_k} & b_{nB_k} \end{array} \right\| \quad \begin{array}{l} (i = 1, \dots, n) \\ (j, k, r = 1, \dots, n-1) \\ (\delta_{rr} = 1, \delta_{rk} = 0 \text{ if } r \neq k), \end{array}$$

which has rank $2n-1$ since the determinant $|Y_1' \quad \xi_{1A_k}|$ is different from zero at $t = t_{10}$. It follows that the matrix (4:13) must have rank $2n-1$ along E . For, if the rank were less than $2n-1$ at a point $t = t'$ on E there would exist constants c, c_k, d_k not all zero such that at that point

$$(4:16) \quad cY_1' + c_k Y_{1A_k} + d_k Y_{1B_k} = 0, \quad cZ_1' + c_k Z_{1A_k} + d_k Z_{1B_k} = 0.$$

Moreover, since the functions in (4:15) satisfy the equations in Theorem 4:5 identically in t, A, B , the columns of the matrix (4:13) are solutions of the linear, homogeneous equations

$$\eta_1' = H_{z_1 y_k} \eta_k + H_{z_1 z_k} \zeta_k, \quad \zeta_1' = -H_{y_1 y_k} \eta_k - H_{y_1 z_k} \zeta_k.$$

The functions

$$\eta_1(t) = cY_1' + c_k Y_{1A_k} + d_k Y_{1B_k}, \quad \zeta_1(t) = cZ_1' + c_k Z_{1A_k} + d_k Z_{1B_k}$$

are also solutions of these equations and since $\eta_1(t_1) = \zeta_1(t_1) = 0$ and $\eta_1'(t_1) = \zeta_1'(t_1) = 0$, it follows that $\eta_1(t) \equiv 0 \equiv \zeta_1(t)$.

Hence, the relations (4:16) would also have to hold at $t = t_0$ which is impossible since it has been proved that the matrix in (4:13) has rank $2n-1$ at that point.

For the Problem of Mayer it will be assumed that the set of multipliers $\lambda_\alpha(t)$ associated with an extremal arc E do not vanish simultaneously. Since the matrix $||\phi_\alpha y_1'|$ has rank m , this assumption is equivalent to the assumption that the functions $z_1(t)$ associated with E do not vanish simultaneously. Hence, not all the constants b_{10} in Theorem 4:1 are zero at the initial element on E . A similar remark applies for the problem of Mayer in parametric form.

Bliss and Hestenes (III, 309) have proved that for the problem of Mayer in non-parametric form the functions $P_1(t, y, z)$ and $L_\alpha(t, y, z)$ in (4:4) satisfy the homogeneity conditions

$$(4:17) \quad P_1(t, y, kz) = P_1(t, y, z), \quad L_\alpha(t, y, kz) = kL_\alpha(t, y, z) \quad (k \neq 0).$$

These conditions imply the following theorem.

THEOREM 4:7. For the problem of Mayer in non-parametric form the Hamilton function H in (4:6) satisfies the condition

$$(4:18) \quad H(t, y, kz) = kH(t, y, z) \quad (k \neq 0),$$

and for proper choice of the parameters the functions in (4:9) satisfy the conditions

$$(4:19) \quad y_1(t, a, kb) = y_1(t, a, b), \quad z_1(t, a, kb) = kz_1(t, a, b) \quad (k \neq 0).$$

The condition (4:18) follows from the first set of conditions in (4:17) and the fact that the function F in (3:1) for the problem of Mayer is identically zero along an arc satisfying the equations $\phi_\alpha = 0$. The last statement in the theorem (III, 310) is a consequence of the conditions (4:17) and

$$F(t, y, y', k\lambda) = kF(t, y, y', \lambda) \quad (k \neq 0).$$

THEOREM 4:8. For the problem of Mayer in parametric form the Hamilton function H in (4:11) satisfies the condition

$$(4:20) \quad H(y, kz) = kH(y, z) \quad (k \neq 0),$$

and for proper choice of parameters the functions in (4:15) satisfy the relations

$$(4:21) \quad Y_1(t, A, kB) = Y_1(t, A, B), \quad Z_1(t, A, kB) = kZ_1(t, A, B) \quad (k \neq 0).$$

From the condition (4:18) for the corresponding non-parametric problem, it follows that the condition (4:20) must hold and that the function b_n in (4:12) satisfies the homogeneity condition $b_n(a, kb_1, \dots, kb_{n-1}, k\chi_0) = kb_n(a, b_1, \dots, b_{n-1}, \chi_0)$. By virtue of the conditions in (4:19), the functions in (4:14) must satisfy the homogeneity relations

$$\begin{aligned} y_1[t, a, kB, b_n(a, kB, k\chi_0)] &= y_1[t, a, B, b_n(a, B, \chi_0)], \\ z_1[t, a, kB, b_n(a, kB, k\chi_0)] &= kz_1[t, a, B, b_n(a, B, \chi_0)]. \end{aligned} \quad (k \neq 0)$$

Since these relations are not affected by replacing a_1 by $\xi_1(A)$, the conditions given in (4:21) must hold for this choice of parameters.

5. The necessary conditions of Weierstrass and Clebsch.

In this section the necessary condition of Weierstrass is given, both in the ordinary and the canonical variables, for problems A and B, as well as the special forms this condition assumes for the corresponding problems of Mayer. Further properties of the Hamilton function are developed and, finally, the condition of Clebsch for the problems A and B is given both in terms of the ordinary variables and the canonical variables.

THEOREM 5:1. The necessary condition of Weierstrass for problem A. At each element t, y, y', χ of a normal minimizing extremal arc E for problem A, the inequality

$$(5:1) \quad \dot{C}(t, y, y', \chi, Y') = F(t, y, Y', \chi) - F(t, y, y', \chi) - (Y'_1 - y'_1)F_{y'_1}(t, y, y', \chi) \geq$$

must hold for all sets $(t, y, Y') \neq (t, y, y')$ interior to the region D_1 and satisfying the equations $\phi_{\alpha} = 0$.

This theorem has been proved by Bliss using a method of Graves (IV, 58).

THEOREM 5:2. The necessary condition of Weierstrass for problem B. At each element y, y', λ of a normal minimizing extremal for problem B, the inequality

$$\hat{C}(y, y', \lambda, Y') = F(y, Y', \lambda) - F(y, y', \lambda) - (Y'_1 - y'_1)F_{y'_1}(y, y', \lambda) \geq 0$$

must hold for all sets $(y, Y') \neq (y, y')$ interior to the region $\mathcal{R}_1$ and satisfying the equations $\phi_\beta = 0$.

This theorem is a consequence of Theorem 5:1 for problem C. Since, along a minimizing extremal arc E' for problem C, the multiplier λ_0 must be equal to zero by Theorem 3:2, the $\hat{C}$ -function (5:1) for this arc is independent of ϕ . Hence, its projection E in the y -space, which is a minimizing extremal for problem B, must satisfy this theorem.

THEOREM 5:3. The necessary condition of Weierstrass for problem A in canonical variables. At each element t, y, z of a normal, non-singular, minimizing extremal arc E for problem A, the inequality

$$(5:2) \quad \hat{C}(t, y, z, Y') = F(t, y, z, Y') + H(t, y, z) - Y'_1 z_1 \geq 0$$

must hold for all sets $(t, y, Y') \neq (t, y, H_z)$ interior to the region D_1 and satisfying the equations $\phi_\alpha = 0$.

This theorem is an immediate consequence of Theorem 5:1 and conditions (4:6), (4:7) and (4:8).

THEOREM 5:4. The necessary condition of Weierstrass for problem B in canonical variables. At each element y, z of a normal, non-singular minimizing extremal arc E for problem B satisfying the equation $H(y, z) = 0$, the inequality

$$(5:3) \quad \mathcal{C}(y, z, Y') = F(y, z, Y') - Y_1' z_1 \geq 0$$

must hold for all sets $(y, Y') \neq (y, H_z)$ interior to the region R_1 and satisfying the equations $\phi_\beta = 0$.

This theorem follows at once from Theorem 5:3 for problem C with the aid of Theorems 3:2, 3:4 and 4:3.

Along an admissible arc for the problem of Mayer in non-parametric form satisfying the equations $\phi_\alpha = 0$, the function F in (3:1) is identically zero. From the homogeneity relation (4:18), it is found by differentiating with respect to k and setting $k = 1$ that

$$(5:4) \quad z_1 H_{z_1} = H.$$

For this problem, condition (5:2) becomes

$$\mathcal{C}(t, y, z, Y') = z_1 (H_{z_1} - Y_1') \geq 0.$$

Similarly, it can be shown that condition (5:3) for the problem of Mayer in parametric form becomes

$$\mathcal{C}(y, z, Y') = -Y_1' z_1 \geq 0.$$

THEOREM 5:5. The necessary condition of Clebsch for problem A. At each element t, y, y', λ of a normal minimizing extremal arc E for problem A, the inequality

$$(5:5) \quad F_{y_1' y_k'}(t, y, y', \lambda) \pi_1 \pi_k \geq 0 \quad (1, k = 1, \dots, n)$$

must be satisfied for every set $(\pi_1, \dots, \pi_n) \neq (0, \dots, 0)$ satisfying the equations

$$(5:6) \quad \phi_{\alpha y_1'}(t, y, y') \pi_1 = 0 \quad (\alpha = 0, 1, \dots, m).$$

If E is non-singular, the equality sign in (5:5) holds only in case

$$(\pi) = (0).$$

The first part of the theorem has been proved by Bliss (IV, 65). If the equality in (5:5) holds for some value $(\pi) \neq (0)$ satisfying the equations (5:6), then from the theory of minima for functions of n variables (IV, 40), it follows that there exist multipliers λ_α such that

$$\begin{aligned} F_{y'_1 y'_k} \pi_k + \lambda_\alpha \phi_{\alpha y'_1} &= 0 \\ \phi_{\alpha y'_k} \pi_k &= 0. \end{aligned}$$

This is possible only in case the determinant of the matrix R in (3:6) is zero.

THEOREM 5:6. The necessary condition of Clebsch for problem B. At each element y, y', λ of a normal minimizing extremal arc E for problem B, the inequality

$$(5:7) \quad F_{y'_1 y'_k}(y, y', \lambda) \pi_1 \pi_k \geq 0 \quad (1, k = 1, \dots, n)$$

must be satisfied for every set $(\pi_1, \dots, \pi_n) \neq (0, \dots, 0)$ satisfying the equations

$$(5:8) \quad \phi_{\beta y'_1}(y, y') \pi_1 = 0 \quad (\beta = 1, \dots, m).$$

If E is non-singular, the equality sign in (5:7) holds only in case $(\pi) = (py')$.

This theorem is a consequence of Theorem 5:5 for problem C which states that for every set π_1 satisfying the equations (5:8) and also $\phi_{0y'_1}(y, y') \pi_1 = 0$, the condition (5:7) must be satisfied. But to every set π_1 satisfying the equations (5:8) there corresponds a set $\pi_1 - py'_1$, with $p = \phi_{y'_j} \pi_j$, satisfying the equations (5:8) and also the equation $\phi_{0y'_1}(\pi_1 - py'_1) = 0$ on account of the homogeneity of the functions ϕ_β and ϕ . For this set $\pi_1 - py'_1$,

$$F_{y_1^i y_k^i} \pi_1 \pi_k = F_{y_1^i y_k^i} (\pi_1 - p y_1^i) (\pi_k - p y_k^i)$$

because of the homogeneity of F . The last expression is ≥ 0 on account of the Clebsch condition for problem C. If E is non-singular, its corresponding arc E' for problem C has the same property, as shown in Section 3. Along such an arc E' , by Theorem 5:5, the above expression is zero only in case $(\pi - p y^i) = (0)$.

In the following lemma and its proof certain properties of the Hamilton function H will be deduced which will be useful in stating and establishing the Clebsch condition in canonical variables.

LEMMA 5:1. Along a non-singular extremal arc E for problem A, the matrix

$$(5:9) \quad \mathcal{L} = \begin{vmatrix} H_{z_j z_k} & L_{\nu} z_j \\ L_{\beta} z_k & 0 \end{vmatrix} \quad \begin{array}{l} (\nu, \beta = 0, 1, \dots, m) \\ (j, k = 1, \dots, n) \end{array}$$

has rank $n+m+1$. Furthermore, the matrix $||L_{\nu} z_j||$ has rank $m+1$ and the matrix $||H_{z_j z_k}||$ has rank $n-m-1$ along E .

Since, by hypothesis, the matrix R in (3:6) has rank $n+m+1$, the linear equations

$$(5:10) \quad \begin{aligned} e_1 &= F_{y_1^i y_j^i} \pi_j + \phi_{\nu} y_1^i \mu_{\nu} & (\nu, \alpha = 0, 1, \dots, m) \\ 0 &= \phi_{\alpha} y_j^i \pi_j & (1, j = 1, \dots, n), \end{aligned}$$

can be solved for π_j and μ_{ν} for arbitrary values of $e_1, \dots, e_n$. In order to state the solutions explicitly, it is observed that if the variables $y_1^i = H_{z_1^i}(t, y, z)$, $\lambda_{\alpha} = L_{\alpha}(t, y, z)$ are substituted in equations (4:3) and the resulting identities are differentiated with respect to z_k , then

$$(5:11) \quad \phi_{\alpha} y_j' H_{z_j z_k} = 0, \quad F_{y_1' y_j'} H_{z_j z_k} + \phi_{\nu} y_1' L_{\nu z_k} = \delta_{1k}$$

where $\delta_{11} = 1$ and $\delta_{1k} = 0$ if $1 \neq k$. Hence,

$$(5:12) \quad \pi_j = H_{z_1 z_j} e_1, \quad \mu_{\nu} = L_{\nu z_1} e_1$$

are the solutions of equations (5:10).

If $e_1 = \phi_{\alpha} y_1'$, then $\pi_j = 0$ by equations (5:12) and (5:11), and it follows from the first group of equations in (5:10) that

$$(5:13) \quad L_{\nu z_k} \phi_{\alpha} y_k' = \delta_{\nu\alpha} \quad (\alpha, \nu = 0, 1, \dots, m).$$

By means of these equations and equations (5:11), it is found that the product of the determinant of the matrix R in (3:6) and that of the matrix $\mathcal{L}$ in (5:9) is

$$\begin{vmatrix} \delta_{1k} & F_{y_1' y_j'} L_{\nu z_j} \\ 0 & \delta_{\alpha\nu} \end{vmatrix} = 1 \quad \begin{matrix} (1, k, j = 1, \dots, n) \\ (\alpha, \nu = 0, 1, \dots, m). \end{matrix}$$

Hence, the matrix $\mathcal{L}$ has rank $n + m + 1$. It follows at once that the matrix $|| L_{\nu z_j} ||$ must have rank $m + 1$ since otherwise the matrix $\mathcal{L}$ could not have rank $n + m + 1$. Similarly, the rank of the matrix $|| H_{z_j z_k} ||$ must be $\geq n - m - 1$. But the first set of equations in (5:11), which represent $m + 1$ linearly independent sets of relations among the rows of this matrix, show that its rank must be $\leq n - m - 1$. Hence, the rank of the matrix $|| H_{z_j z_k} ||$ is exactly $n - m - 1$.

THEOREM 5:7. The necessary condition of Clebsch for problem A in canonical variables. At each element (t, y, z) of a normal non-singular minimizing extremal arc E for problem A, the inequality

$$H_{z_1 z_k} e_1 e_k > 0 \quad (1, k = 1, \dots, n)$$

must be satisfied for every set $(e_1, \dots, e_n) \neq (0, \dots, 0)$ satisfying the equations

$$(5:14) \quad L_{\nu} z_1 e_1 = 0 \quad (\nu = 0, 1, \dots, m).$$

Let e_1 be a solution of equations (5:14). If π_1 are defined by the expressions $H_{z_1 z_k} e_k$, it follows from the first group of equations in (5:11) that π_1 satisfy the equations $\phi_{\alpha y_1'} \pi_1 = 0$. Conversely, let π_1 be a solution of the equations $\phi_{\alpha y_1'} \pi_1 = 0$. Then if e_1 are defined by the expressions $F_{y_1' y_j'} \pi_j$, it is seen with the aid of the equations (5:10) and their solutions (5:12) that e_1 must satisfy the equations (5:14). Furthermore, the numbers π_1 are all zero if and only if the corresponding numbers e_1 are all zero, since the determinant of the coefficients in the equations (5:10) is different from zero along a non-singular arc. It should be noted that equations (5:12) and (5:11) show that $e_1 \neq h_{\alpha} \phi_{\alpha y_1'}$ since otherwise $(\pi) = (0)$. This condition on the sets (e) is implied by the equations (5:14) on account of the conditions (5:13). It is now seen with the aid of equations (5:10) and their solutions (5:12), that for every set $(\pi) \neq (0)$ satisfying the equations $\phi_{\alpha y_1'} \pi_1 = 0$, there corresponds a set $(e) \neq (0)$ satisfying the equations (5:14), and conversely, for which

$$H_{z_1 z_k} e_1 e_k = F_{y_1' y_k'} \pi_1 \pi_k.$$

The present theorem now follows from Theorem 5:5.

THEOREM 5:8. The necessary condition of Clebsch for problem B in canonical variables. At each element (y, z) of a normal non-singular minimizing extremal arc E for problem B, the inequality

$$(5:15) \quad H_{z_1 z_k} e_1 e_k > 0 \quad (1, k = 1, \dots, n)$$

must be satisfied for every set $(e_1, \dots, e_n) \neq (p\phi_{y_1}, \dots, p\phi_{y_n})$
satisfying the equations

$$(5:16) \quad L_{\beta z_1} e_1 = 0, \quad (\beta = 1, \dots, m)$$

where the function ϕ is the function appearing in the function ϕ_0
defined by equation (1:5).

This theorem is a consequence of Theorem 5:7 for problem C which states that for every set $(e_1, \dots, e_n) \neq (0, \dots, 0)$ satisfying equations (5:16) and also $L_{oz_1} e_1 = 0$, the condition (5:15) must be satisfied. But to every set e_1 satisfying the equations (5:16) there corresponds a set $e_1 - p\phi_{y_1}$ with $p = L_{oz_k} e_k$, satisfying the conditions (5:16) and also the equation $L_{oz_1}(e_1 - p\phi_{oy_1}) = 0$ on account of the condition (5:13). For this set $e_1 - p\phi_{oy_1}$

$$H_{z_1 z_k} e_1 e_k = H_{z_1 z_k} (e_1 - p\phi_{oy_1})(e_k - p\phi_{oy_k})$$

because of the first group of equations in (5:11). The right member of the last expression must be positive for $(e - p\phi_{oy}) \neq (0)$ by Theorem 5:7 for problem C.

6. Mayer fields for problem A. A Mayer field, as defined by Bliss, is a region $\mathcal{H}'$ in ty -space containing only interior points and having associated with it a set of slope functions and multipliers $p_1(t, y)$, $\chi_\alpha(t)$, having continuous first partial derivatives in $\mathcal{H}'$, defining elements $[t, y, p(t, y)]$ which lie in the region D_1 and satisfy the equations $\phi_\alpha = 0$ for every (t, y) in $\mathcal{H}'$, and making the integral

$$I^* = \int (F - p_1 F_{y_1}) dt + F_{y_1} dy_1$$

independent of the path in $\mathcal{F}'$. The arguments of F and its derivatives in the integrand of I^* are $[t, y, p(t, y), \lambda(t, y)]$.

Bliss has shown that the solutions of the equations $y_1' = p_1(t, y)$ are extremals with multipliers $\lambda_\alpha[t, y(t)]$, called the extremals of the field (IV, 103). It should be noted that for the problem of Mayer the function $F(t, y, p, \lambda)$ vanishes identically in $\mathcal{F}'$ and the integral I^* has the value zero along every extremal of the field.

Let $\bar{D}$ be a region of points (t, y, y', λ) with (t, y, y') interior to the region D_1 described in Section 2 and so related to a region N of points (t, y, q) that to each point (t, y, q) interior to N there corresponds a unique solution (t, y, y', λ) of equations (4:3) with (t, y, y', λ) interior to $\bar{D}$ and the determinant of the matrix R in (3:6) different from zero. The equations (4:3) with $z_1 = q_1$ have as before single-valued solutions with continuous second derivatives over the interior of N , and every solution $y_1(t)$, $z_1(t) = q_1(t)$ of equations (4:8) in this region defines an extremal. A set (t, y, q) interior to N will be called an admissible set.

A second definition of a Mayer field is a region $\mathcal{F}$ of ty -space containing only interior points with which are associated functions $q_1(t, y)$ having continuous first partial derivatives in $\mathcal{F}$, defining elements $[t, y, q(t, y)]$ interior to the region N for every (t, y) in $\mathcal{F}$, and making the integral

$$I^* = \int q_1(t, y) dy_1 - H[t, y, q(t, y)] dt$$

independent of the path in $\mathcal{F}$.

THEOREM 6:1. If the integral I^* is independent of the path in $\mathcal{F}$, the equations

$$(6:1) \quad -\partial H / \partial y_1 = \partial q_1 / \partial t, \quad \partial q_1 / \partial y_k = \partial q_k / \partial y_1 \\ (1, k = 1, \dots, n)$$

hold identically in the variables (t, y) .

From the hypothesis of the theorem, it follows that every arc in $\mathcal{F}$ is a minimizing arc for the integral I^* and hence must satisfy the corresponding Euler-Lagrange differential equations for I^* . In order that this be true, equations (6:1) must hold.

Along an admissible arc $y_1 = y_1(t)$ in $\mathcal{F}$, the identity

$$(6:2) \quad \partial q_1 / \partial t + \partial H / \partial y_1 = dq_1 / dt + H_{y_1} + q_1 y_k (H_{z_k} - y_k')$$

is readily seen to be true since

$$\partial H / \partial y_1 = H_{y_1} + H_{z_k} q_k y_1 = H_{y_1} + H_{z_k} q_1 y_k, \quad dq_1 / dt = q_1 t + q_1 y_k y_k'.$$

THEOREM 6:2. Every solution $y_1(t)$ in the field $\mathcal{F}$ of the equations

$$(6:3) \quad dy_1 / dt = H_{z_1} [t, y, q(t, y)]$$

is an extremal with $z_1(t) = q_1 [t, y(t)]$.

The sets $[t, y, q(t, y)]$ are interior to N for (t, y) in $\mathcal{F}$, and equations (6:1) and (6:2) show that along an arc satisfying equations (6:3), the equation

$$dq_1 / dt + H_{y_1} = 0$$

must hold. If $z_1(t)$ is defined to be $q_1 [t, y(t)]$, then $y_1(t)$ and $z_1(t)$ satisfy the equations

$$dy_1 / dt = H_{z_1}(t, y, z), \quad dz_1 / dt = -H_{y_1}(t, y, z),$$

which are the canonical equations of the extremals. The arcs

satisfying the equations (6:3) with $z_1(t) = q_1[t, y(t)]$ are called the extremals of the field.

THEOREM 6:3. Through each point of the field $\mathcal{F}$ there passes one and only one extremal of the field.

This theorem follows from the fact that the equations (6:3) are of first order.

A Mayer field as defined by Bliss is a field according to the second definition, provided that the sets $[t, y, q(t, y)]$ defined by points (t, y) in $\mathcal{F}'$, where $q_1(t, y) = F_{y_1}[t, y, p(t, y), \chi(t, y)]$ are interior to the region N . Every field of the second type is a Mayer field as defined by Bliss with $p_1(t, y) = H_{z_1}[t, y, q(t, y)]$ and $\chi_\alpha(t, y) = L_\alpha[t, y, q(t, y)]$.

7. Transversal surfaces of a Mayer field for problem A.

Let $\mathcal{F}$ be a field defined by functions $q_1(t, y)$. An n -space $W(t, y) = \text{constant}$ is called a transversal surface of the field if at each point of the n -space the equation

$$q_1(t, y) dy_1 - H[t, y, q(t, y)] dt = 0$$

is satisfied by every set of differentials dt, dy_1 which satisfy the equation

$$W_{y_1} dy_1 + W_t dt = 0.$$

It is evident that a necessary and sufficient condition that an n -space $W(t, y) = \text{constant}$ be a transversal surface of a field $\mathcal{F}$ is that at each point on the n -space in $\mathcal{F}$

$$(7:1) \quad W_{y_1} = k q_1, \quad W_t = -k H$$

where k is a proportionality factor.

THEOREM 7:1. In every field $\mathcal{F}$ the Hilbert Integral defines a function $W(t,y)$ having continuous second partial derivatives in $\mathcal{F}$, satisfying the Hamilton-Jacobi partial differential equation

$$(7:2) \quad W_t + H(t,y,W_y) = 0,$$

and such that the surfaces of the family $W(t,y) = \text{constant}$ are the transversal surfaces of the field. Conversely, let $W(t,y)$ be a function which has continuous second partial derivatives and defines only admissible elements $(t,y,q) = (t,y,W_y)$ in a region $\mathcal{F}$ of ty -space. Then if W satisfies equation (7:2), the region $\mathcal{F}$ is a field with

$$(7:3) \quad q_1(t,y) = W_{y_1},$$

and the surfaces $W(t,y) = \text{constant}$ are the transversal surfaces of the field.

In a field $\mathcal{F}$ determined by a set of functions $q_1(t,y)$, the function

$$(7:4) \quad W(t,y) = \int_{(t_0,y_0)}^{(t,y)} q_1(t,y) dy_1 - H[t,y,q(t,y)] dt,$$

where (t_0,y_0) is a fixed point in $\mathcal{F}$, has the properties described in the first part of the theorem since its derivatives are given by equations (7:1) with $k = 1$. The equations (7:1) with $k = 1$ are clearly equivalent to equation (7:2).

Conversely, every solution W of equation (7:2) with the properties described in the theorem defines functions q_1 by means of equations (7:3) which with W satisfy the equations $W_t = -H(t,y,q)$, $q_1 = W_{y_1}$. The Hilbert integral I^* formed with the functions $q_1 = W_{y_1}$ is the integral of dW and hence is independent of the path.

Thus the region $\mathcal{F}$ is a field with functions $q_1 = W_{y_1}$, according to the second definition in Section 6.

The following well-known properties of a field are easily established for a field $\mathcal{F}$. In a field, $I^*(C_{12}) = W(t, y)|_1^2$ for any two points 1 and 2. Therefore, for every arc C_{12} on a transversal surface, $I^*(C_{12})$ is identically zero. From this it follows that, if the extremals of the field do not lie on the transversal surfaces, then two transversal surfaces intercept arcs on the extremals of the field along which the integral $\int (z_1 H_{z_1} - H) dt$ has a constant value.

8. Construction of fields and relations between extremals and a complete integral of the Hamilton-Jacobi equation for problem A. In this section conditions are given under which it is possible to construct an n -parameter family of fields from a $2n$ -parameter family of extremals. The complete integral of the Hamilton-Jacobi equation is determined with the aid of a $2n$ -parameter family of extremals, and conditions which this integral must satisfy if some of the parameters do not appear in the functions y_1 defining the extremals are derived. Finally it is shown how to determine a $2n$ -parameter family of extremals from a complete integral, and conditions are given which imply the non-appearance of some of the parameters in the functions y_1 defining the extremals.

THEOREM 8:1. Suppose that a non-singular extremal arc E for problem A is a member for values $t_{10} \leq t \leq t_{20}$, $a_{10}, \dots, a_{n0}$, $b_{10}, \dots, b_{n0}$ of a $2n$ -parameter family of extremals defined by functions of the form

$$(8:1) \quad y_1(t, a_1, \dots, a_n, b_1, \dots, b_n), \quad z_1(t, a_1, \dots, a_n, b_1, \dots, b_n)$$

which are such that the functions y_1, y_{1t}, z_1, z_{1t} have continuous

first derivatives in a neighborhood $\mathcal{B}$ of the values (t, a, b) belonging to E . Suppose further that for each set (t, a, b) in $\mathcal{B}$ the determinants

$$(8:2) \quad \begin{vmatrix} y_{1a_k} \\ y_{1a_k} \end{vmatrix}, \quad \begin{vmatrix} y_{1a_k} & y_{1b_k} \\ z_{1a_k} & z_{1b_k} \end{vmatrix}$$

are different from zero and the equations

$$(8:3) \quad y_{1a_k} z_{1a_j} - y_{1a_j} z_{1a_k} = 0 \quad (1, k, j = 1, \dots, n)$$

hold identically. Then for fixed values of (b) in $\mathcal{B}$ the family of extremals defined by the functions (8:1) form a Mayer field over a neighborhood $\mathcal{F}$ of the points (t, y) on E . Furthermore, the functions $q_1(t, y, b)$ for each field have the determinant

$$(8:4) \quad \begin{vmatrix} q_{1b_k} \end{vmatrix} \quad (1, k = 1, \dots, n)$$

different from zero. The functions $q_1(t, y, b)$ are defined by the equations

$$(8:5) \quad q_1(t, y, b) = z_1[t, a(t, y, b), b],$$

and the functions $a_1(t, y, b)$ are found by solving the equations

$$(8:6) \quad y_1 = y_1(t, a, b).$$

Since the determinant $|y_{1a_k}|$ is different from zero for (t, a, b) in $\mathcal{B}$, the system of equations (8:6) has a unique solution $a_1 = a_1(t, y, b)$ for (b) in $\mathcal{B}$ and (t, y) in a neighborhood $\mathcal{F}$ of the points (t, y) on E ; the functions $a_1(t, y, b)$ have continuous second derivatives for (b) in $\mathcal{B}$ and (t, y) in $\mathcal{F}$ since the right members of equations (8:6) have such derivatives. It follows from conditions (8:3) that for each fixed set (b) in $\mathcal{B}$ the family

of extremals defined by the functions (8:1) form a Mayer field over the region $\mathcal{F}$ with functions $q_1(t, y, b)$ defined as in (8:5) (I, 595). There is a family of such fields depending upon the n parameters (b) .

In each field the determinant (8:4) must be different from zero. For, the differentials of the equations (8:5) and the identities $y_1 = y_1[t, a(t, y, b), b]$ with respect to b_j satisfy the equations

$$q_{1b_j} db_j = z_{1a_k} da_k + z_{1b_j} db_j, \quad (i, j, k = 1, \dots, n)$$

$$0 = y_{1a_k} da_k + y_{1b_j} db_j,$$

where $da_k = a_{kb_j} db_j$. If the determinant (8:4) were equal to zero at a point (t, y) in $\mathcal{F}$ and (b) in $\mathcal{B}$, there would exist values db_j not all zero satisfying the equation $q_{1b_j} db_j = 0$. For such values of db_j , the second determinant in (8:2) would have the value zero which is contrary to hypothesis.

The following theorem (II, 268; VIII, 146) indicates conditions under which an extremal satisfying the hypotheses of the previous theorem exists.

THEOREM 8:2. Let E be a normal non-singular extremal for problem A for which there exists a conjugate system η_{1k}, ζ_{1k} of solutions of the canonical accessory equations

$$(8:7) \quad \eta'_1 = H_{z_1 y_k} \eta_k + H_{z_1 z_k} \zeta_k, \quad \zeta'_1 = -H_{y_1 y_k} \eta_k - H_{y_1 z_k} \zeta_k$$

whose determinant $|\eta_{1k}(t)|$ is different from zero along E . Then E is a member of a $2n$ -parameter family of extremals of the type described in Theorem 8:1.

The equations (8:7) are derived by Bliss in his lectures on the Problem of Bolza (IV, 75). To establish the theorem let

$$(8:8) \quad \begin{aligned} y_1 &= Y_1(t, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n), \\ z_1 &= Z_1(t, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \end{aligned}$$

be a $2n$ -parameter family of extremals containing E for values $\alpha_1 = \alpha_{10}, \beta_1 = \beta_{10}, t_{10} \leq t \leq t_{20}$ and such that the parameters α_1, β_1 are the values of the functions Y_1, Z_1 at $t=t_{10}$, the functions Y_1, Y_{1t}, Z_1, Z_{1t} having continuous first and second partial derivatives in a neighborhood of the values (t, α, β) belonging to E . The $2n$ -parameter family

$$(8:9) \quad y_1 = y_1(t, a_1, \dots, a_n, b_1, \dots, b_n), \quad z_1 = z_1(t, a_1, \dots, a_n, b_1, \dots, b_n),$$

obtained from the family (8:8) by substituting for (α, β) the functions $\alpha(a), \beta(a, b)$ defined by the equations

$$(8:10) \quad \alpha_1(a) = \alpha_{10} + \eta_{1k}(t_{10})a_k, \quad \beta_1(a, b) = \beta_{10} + \zeta_{1k}(t_{10})a_k + b_1,$$

contains E for $(a, b) = (0, 0)$ and can be shown to have the properties described in Theorem 8:1. For, by differentiating the identities

$$y_1(t_{10}, a, b) = \alpha_1(a), \quad z_1(t_{10}, a, b) = \beta_1(a, b)$$

with respect to a_k, b_k it is found that at $t = t_{10}$,

$$(8:11) \quad y_{1a_k} = \eta_{1k}(t_{10}), \quad z_{1a_k} = \zeta_{1k}(t_{10}), \quad y_{1b_k} = 0, \quad z_{1b_k} = \delta_{1k}$$

where $\delta_{11} = 1$ and $\delta_{1k} = 0$ if $1 \neq k$. Hence, along E

$$y_{1a_k} = \eta_{1k}(t), \quad z_{1a_k} = \zeta_{1k}(t)$$

since along E the right and left members of these equations are solutions of equations (8:7) with the same initial values at $t = t_{10}$. The determinant $|y_{1a_k}|$ is therefore different from zero along E . It follows that the first n equations in (8:9) have unique solutions

$a_1 = a_1(t, y, b)$ which are continuous and have continuous first and second derivatives for all (t, y, b) such that (t, y) is in a neighborhood $\mathcal{F}$ of points (t, y) on E and (b) in a neighborhood $\mathcal{B}$ of $(b) = (0)$. For each fixed set (b) in $\mathcal{B}$ and (a) in a neighborhood of (0) ,

$$y_{1a_k} z_{1a_j} - y_{1a_j} z_{1a_k} = \eta_{1k}(t_{10}) S_{1j}(t_{10}) - \eta_{1j}(t_{10}) S_{1k}(t_{10})$$

on the hyperplane $t = t_{10}$. The right members of these equations have the value zero since η_{1k}, S_{1k} is a conjugate system of solutions of the equations (8:7) (IV, 80). Since the left members of these equations are constant along each extremal of the family (8:9) for (t, a, b) in a neighborhood of the set (t, a, b) on E (I, 595), it follows that condition (8:3) of Theorem 8:1 is satisfied. With the aid of equations (8:11) it is seen that the second determinant in (8:2) is different from zero at $t = t_{10}$ on E and hence must be different from zero along E (IV, 37). Therefore, there is at least a neighborhood of the values (t, a, b) belonging to E in which the $2n$ -parameter family of extremals (8:9) satisfies the hypotheses of Theorem 8:1.

A complete integral of the Hamilton-Jacobi partial differential equation will now be determined with the aid of a $2n$ -parameter family of extremals.

THEOREM 8:3. If a non-singular extremal arc E for problem A is a member of a $2n$ -parameter family of extremals having the properties described in Theorem 8:1, then there is a neighborhood of E in which, with an arbitrary additive constant, the function

$$W(t, y, b) = \int_{(t_0, y_0)}^{(t, y)} q_1(t, y, b) dy_1 - H(t, y, q) dt$$

with q_1 defined by equations (8:5), is a complete integral of the

Hamilton-Jacobi partial differential equation $W_t + H(t, y, W_y) = 0$
and has the determinant $|W_{y_1 b_k}|$ ($i, k = 1, \dots, n$) different from
zero.

This theorem follows directly from Theorem 8:1 and the first part of Theorem 7:1.

The theorem of Jacobi on the determination of the extremals from a complete integral of the Hamilton-Jacobi equation will now be given.

THEOREM 8:4. If $W(t, y_1, \dots, y_n, b_1, \dots, b_n)$ has con-
tinuous second partial derivatives with the determinant

$$(8:12) \quad \left| W_{b_1 y_k} \right| \quad (i, k = 1, \dots, n)$$

different from zero for all points (t, y, b) satisfying conditions
of the form (t, y) in a region $\mathcal{F}$, (b) in a region $\mathcal{G}$, and the sets
 $(t, y, q) = (t, y, W_y)$ corresponding to these arguments admissible,
and if furthermore W satisfies the Hamilton-Jacobi partial differ-
ential equation

$$(8:13) \quad W_t + H(t, y, W_y) = 0$$

identically in (t, y, b) , then the equations

$$(8:14) \quad W_{b_1}(t, y, b) = W_{b_1}(t_0, a, b)$$

define a $2n$ -parameter family of extremals

$$y_1(t, t_0, a, b), \quad z_1(t, t_0, a, b)$$

with properties similar to those described in Theorem 4:1, the
functions $z_1(t, t_0, a, b)$ being defined by the equations

$$(8:15) \quad z_1 = W_{y_1}[t, y(t, t_0, a, b), b].$$

Since the function W satisfies the equation (8:13) identically in (t, y, b) , it follows that W must satisfy the equations

$$(8:16) \quad W_{tb_1} + H_{z_k} W_{y_k} b_1 = 0, \quad W_{ty_1} + H_{y_1} + H_{z_k} W_{y_k} y_1 = 0,$$

found by differentiating the equations (8:14) with respect to b_1 and y_1 , respectively.

It will now be shown that every admissible arc E interior to $\mathcal{H}$, having no corners, and satisfying the equations (8:14) for particular constant values a_{10} , b_{10} is an extremal. For, the functions $y_1(t)$ ($t_1 \leq t \leq t_2$) defining E with the functions $z_1(t)$ defined by means of (8:15) satisfy the equations

$$(8:17) \quad W_{b_1 t} + W_{b_1 y_k} y_k' = 0, \quad z_1' = W_{y_1 t} + W_{y_1 y_k} y_k',$$

found by differentiating the equations (8:14) and (8:15) with respect to t . The equations (8:16) and (8:17) show, with the aid of the condition (8:12), that the functions $y_1(t)$, $z_1(y)$ satisfy the canonical equations (4:8) and hence that E is an extremal.

The values (t, y, t_0, a, b) belonging to E form a set satisfying equations (8:14) on which the determinant (8:12) is different from zero. Hence, equations (8:14) have solutions $y_1 = y_1(t, t_0, a, b)$ which, substituted in (8:15), define functions $z_1 = z_1(t, t_0, a, b)$. For a fixed value of t_0 , these functions define a $2n$ -parameter family of extremals

$$(8:18) \quad y_1(t, a, b), \quad z_1(t, a, b)$$

including E for the parameter values $a_1 = a_{10}$, $b_1 = b_{10}$ and having continuous partial derivatives of at least the first order in a neighborhood of these values since the left members of equations (8:14) and the right members of equations (8:15) have such

derivatives. Since, from the discussion above, the functions in (8:18) satisfy the canonical equations (4:8), it follows that y_{1t} , z_{1t} also have continuous first derivatives. By substituting the functions (8:18) in equations (8:14) and (8:15) and differentiating the resulting identities with respect to a_j , b_j , it is readily seen that the product of the determinants

$$\begin{vmatrix} w_{b_1 y_k} & 0 \\ w_{y_1 y_k} & -\delta_{1k} \end{vmatrix} \begin{vmatrix} y_{ka_j} & y_{kb_j} \\ z_{ka_j} & z_{kb_j} \end{vmatrix} \quad (i, j, k = 1, \dots, n)$$

is the determinant

$$\begin{vmatrix} w_{b_1 y_j}(t_0, a, b) & w_{b_1 b_j}(t_0, a, b) - w_{b_1 b_j}(t, y, b) \\ 0 & -w_{y_1 b_j}(t, y, b) \end{vmatrix}$$

and hence that the value of the functional determinant of $\{y, z\}$ with respect to (a, b) has rank $2n$. This establishes the final property of the family (4:9) as belonging also to the family (8:18).

For certain problems of Bolza some of the parameters (b) do not appear in the first n functions (8:1) defining a $2n$ -parameter family of extremals for the problem. For the problem of Mayer, for example, at least one of the parameters (b) is non-essential in the first n functions in (8:1) as shown by the homogeneity relations (4:19). A similar condition holds in the case of the problem of Lagrange with finite side conditions transformed by differentiation of the side conditions into a problem of Lagrange with differential equations as side conditions (VI, 11). It should be noted that in the first of these problems the order of abnormality (IV, 52) is at least 1 and in the second it is equal to $m+1$. In the following

theorem a condition is given which is necessary and sufficient for the non-appearance of some of the parameters (b) in the first n functions in (8:1).

THEOREM 8:5. If the last s of the parameters (b) do not appear in the functions $y_1(t, a, b)$ in (8:1) of Theorem 8:1, where s is one of the numbers $1, \dots, m+1$, then the function W described in Theorem 8:3 satisfies the additional equations

$$(8:19) \quad H_{z_k z_1} W_{y_1} b_\rho = 0 \quad (\rho = n-s+1, \dots, n; 1, k=1, \dots, n).$$

Conversely, if the function $W(t, y, b)$ having the properties described in Theorem 8:4 has continuous third derivatives and also satisfies the equations (8:19), then the last s of the parameters (b) do not appear in the functions $y_1(t, a, b)$ in (8:18) defining the extremals.

To establish the first statement in the theorem it should be noted that if the last s of the parameters $b_1, \dots, b_n$ do not appear in the functions $y_1(t, a, b)$ in (8:1), then the functions $a_1(t, y, b)$ described in Theorem 8:1 are independent of these parameters also. Differentiating the identity

$$W_{y_1}(t, y, b) = z_1[t, a(t, y, b), b]$$

for b_ρ , it is found that $W_{y_1} b_\rho = z_{1b_\rho}$. Moreover, from the relation

$$y_1'(t, a, b) = H_{z_1}[t, y(t, a, b), z(t, a, b)]$$

one finds by differentiating for b_ρ that $H_{z_1 z_k} z_{kb_\rho} = 0$ since the functions $y_1(t, a, b)$, $y_1'(t, a, b)$ are independent of b_ρ . Equations (8:19) follow easily from these relations.

To prove the last part of the theorem let $W(t, y, b)$ be a solution of the Hamilton-Jacobi equation (8:13) as described in Theorem 8:4. By the latter part of Theorem 7:1, the region $\mathcal{J}$ in

ty-space for each fixed set (b) in $\mathcal{B}$ is a field with functions $q_1(t, y, b) = W_{y_1}(t, y, b)$ and with extremals of each field defined by the functions (8:18) in which the parameters a_1 are assumed to be the values of the functions y_1 at $t = t_0$. The invariant integral (7:4) from a fixed point (t_0, a) to a variable point (t, y) in $\mathcal{F}$ is given by

$$\int_{(t_0, a)}^{(t, y)} (q_k dy_k - H dt) = W(t, y, b) - W(t_0, a, b).$$

It follows that

$$\begin{aligned} (8:20) \quad W_{b_1}(t, y, b) &= W_{b_1}(t_0, a, b) + \int_{(t_0, a)}^{(t, y)} q_{kb_1} (dy_k - H_{z_k} dt), \\ W_{b_1 b_j}(t, y, b) &= W_{b_1 b_j}(t_0, a, b) + \int_{(t_0, a)}^{(t, y)} [q_{kb_1 b_j} (dy_k - H_{z_k} dt) \\ (8:21) \quad &- H_{z_k z_h} q_{kb_1} q_{hb_j} dt] \quad (i, j, k, h = 1, \dots, n). \end{aligned}$$

If the functions $y_1(t, a, b)$ are substituted for y_1 in the equations (8:14) and the resulting identities are differentiated for b_ρ , it is found with the aid of the identities obtained by making the same substitution in equations (8:21), (8:19) and $W_{y_1 b_\rho} = q_{1b_\rho}$ that

$$W_{b_1 y_j y_j b_\rho} = - \int_{(t_0, a)}^{(t, y)} q_{kb_1 b_\rho} (dy_k - H_{z_k} dt).$$

In view of the fact that the functions $y_1(t, a, b)$ define an extremal of the field passing through the point (t_0, a) , the invariant line integral above may be evaluated along this extremal. When this is done, it is found to have the value zero. Accordingly, one obtains the relations $W_{b_1 y_j y_j b_\rho} = 0$. Since the determinant in (8:12) is different from zero for (t, y) in $\mathcal{F}$ and (b) in $\mathcal{B}$, it follows that the functions $y_1(t, a, b)$ in (8:18) must, therefore, have the form stated in the theorem.

9. Fields for the problem of Mayer in non-parametric form.

For this problem, as stated in Theorem 4:7, the Hamilton function $H(t, y, z)$ is homogeneous of order one in z . Because of this fact, the fields for this problem have certain peculiar properties which will be discussed in this section. It will be recalled also that it was shown in Section 4 that the multipliers $\lambda_\alpha(t)$ associated with an extremal arc E for this problem vanish simultaneously if and only if the functions $z_1(t)$ associated with E have this property.

THEOREM 9:1. The integral I^* has the value zero along every extremal of a field $\mathcal{H}$. Furthermore, every extremal of a field lies on one of its transversal surfaces $W(t, y) = \text{constant}$ and the values W_{y_1} are not all zero.

Along each extremal of a field, the integrand of the invariant integral I^* is equal to the expression $z_1 H_{z_1} - H$ which is identically zero on account of the condition (5:4). It follows that each extremal of the field lies on the transversal surface $W(t, y) = W(t_0, y_0)$ defined by the function (7:4) where (t_0, y_0) is a fixed point on the extremal. The values W_{y_1} are not all zero since they are the values of the functions z_1 belonging to the extremals of the field.

THEOREM 9:2. Let E be a non-singular extremal arc for the problem of Mayer having the properties described in Theorem 8:1. Suppose further that the functions $y_1(t, a, b)$, $z_1(t, a, b)$ in (8:1) satisfy the homogeneity conditions (4:19). Then the functions $q_1(t, y, b)$ defined in (8:5) satisfy the conditions

(9:1)

$$q_1(t, y, kb) = kq_1(t, y, b)$$

($k \neq 0$).

This theorem follows from the fact that the functions $a_1(t, y, b)$ described in Theorem 8:1 would, under the hypotheses of this theorem, satisfy the conditions $a_1(t, y, kb) = a_1(t, y, b)$ ($k \neq 0$).

THEOREM 9:3. Let E be a non-singular extremal for the problem of Mayer having the properties described in Theorem 8:2. Then E is a member of a 2n-parameter family of extremals of the type described in Theorem 9:2.

Since the parameters α_1, β_1 in the functions $Y_1(t, \alpha, \beta), Z_1(t, \alpha, \beta)$ in (8:8) are the initial values of Y_1, Z_1 at $t = t_0$ these functions satisfy the relations (4:19) and not all the parameters β_{10} belonging to E are zero (III, 310). In order to preserve this property for the functions $y_1(t, a, b), z_1(t, a, b)$ in (8:9), the expressions for β_1 in (8:10) are replaced by the following expressions which are homogeneous of order one in (b):

$$\beta_r = S_{rk}(t_0) a_k b_n + b_r, \quad \beta_n = S_{nk}(t_0) a_k b_n + \beta_{n0} b_n, \quad (r = 1, \dots, n-1)$$

where it is assumed that the functions y_1, z_1 in (8:8) have been renumbered, if necessary, so that $\beta_{n0} \neq 0$ and the functions $\gamma_1(a)$ in (8:10) are unchanged. The family (8:9) now contains the extremal arc E for $a_1 = 0, b_r = \beta_{r0}, b_n = 1$, and the functions $y_1(t, a, b), z_1(t, a, b)$ in (8:9) satisfy the relations (4:19). The proof given for Theorem 8:2 with a slight modification in the last set of equations in (8:11) is valid for the parameters (β) replaced by the expressions given above.

THEOREM 9:4. If E is a non-singular extremal arc for the problem of Mayer satisfying the hypotheses of Theorem 9:2, then the function W described in Theorem 8:3 without the additive constant can be put into the form $b_n U(t, y_1, \dots, y_n, c_1, \dots, c_{n-1})$, and the number s described in the first part of Theorem 8:5 is at

least equal to one. Conversely, if a function $U(t, y_1, \dots, y_n, c_1, \dots, c_{n-1})$ has continuous second partial derivatives with the determinant

$$\begin{vmatrix} U_{y_1} \\ U_{y_1 c_j} \end{vmatrix} \quad (i = 1, \dots, n; j = 1, \dots, n-1)$$

different from zero for all points (t, y, c) satisfying conditions of the form (t, y) in a region $\mathcal{H}$, (c) in a region C , and the sets $(t, y, c) = (t, y, U_y)$ corresponding to these arguments admissible, and if U satisfies the Hamilton-Jacobi equation (8:13) identically in t, a, c , then $W = b_n U(t, y_1, \dots, y_n, c_1, \dots, c_{n-1})$ for $b_n \neq 0$ is a function possessing the properties of the function W described in Theorem 8:4. If U also has continuous third derivatives, then the number s described in the last part of Theorem 8:5 is at least equal to one.

The first part of the theorem follows from the conditions (4:18) and (9:1). The final statement in the theorem is a consequence of the homogeneity of the function H given in (4:18). To prove $s \geq 1$, we use the identities $z_1 H_{z_1 z_j} = 0$, $H_{z_1 z_j}(t, y, kz) = (1/k) H_{z_1 z_j}(t, y, z)$, obtained by differentiating the equation (4:18) first with respect to z_1 and the resulting equation with respect to k and with respect to z_j . With the help of these identities and $z_1 = U_{y_1}$, it is found that

$$H_{z_1 z_k}(y, W_y) W_{y_k} b_n = \frac{1}{b_n} H_{z_1 z_k}(y, U_y) U_{y_k} = 0.$$

The number s is accordingly at least one by Theorem 8:5.

THEOREM 9:5. Each transversal surface $W(t, y) = c$ of a field $\mathcal{H}$, determined by functions $q_1(t, y)$, contains an $(n-1)$ -parameter family of extremals of the field defined by functions

of the form

$$(9:2) \quad y_1(t, a_1, \dots, a_{n-1}, c), \quad z_1(t, a_1, \dots, a_{n-1}, c).$$

At each point on the transversal surface, the equations

$$(9:3) \quad q_1(t, y) y_{1a_k} = 0 \quad (i = 1, \dots, n; k = 1, \dots, n-1)$$

are identically satisfied by the functions $y_1(t, a, c)$ in (9:2).

Let the extremals of the field be given by the equations

$$(9:4) \quad y_1 = Y_1(t, a_1, \dots, a_n), \quad z_1 = Z_1(t, a_1, \dots, a_n)$$

where (a) lie in a region $\mathcal{A}$ and are the values of Y_1 at some point $t = t_0$ on the interval $t_1 \leq t \leq t_2$. Then, by the first part of Theorem 7:1, the transversal surfaces are determined by the equation $W(t, y) = \text{constant}$, where

$$(9:5) \quad W(t, y) = \int_{(t_0, y_0)}^{(t, y)} q_1(t, y) dy_1 - H[t, y, q(t, y)] dt.$$

In view of Theorem 9:1 one has the identity

$$W[t, y(t, a)] = W(t_0, a).$$

Since the values W_{y_1} are not all zero, we can infer from the equation $W(t_0, a) = c$, that the parameters $a_1, \dots, a_n$ can be replaced by new parameters one of which is the constant c . We may denote these new parameters by $a_1, \dots, a_{n-1}, c$ and hence express the extremals of the field by functions of the form 9:2. The equation (9:3) is obtained by differentiating the identity

$$W[t, y(t, a_1, \dots, a_{n-1}, c)] = c$$

with respect to a_k and replacing W_{y_1} by its value $q_1(y)$. This proves Theorem 9:5.

10. Mayer fields and transversal surfaces for problem B.

A Mayer field as defined by Bliss, is a region $\mathcal{H}'$ of y -space containing only interior points and with which there is associated a set of slope-functions and multipliers $p_1(y)$, $\lambda_\beta(y)$ having continuous first partial derivatives in $\mathcal{H}'$, defining elements (y, p) which lie in the region $\mathcal{R}_1$ and satisfy the equations $\phi_\beta = 0$ for every (y) in $\mathcal{H}'$, and making the integral $I^* = \int_{y_1} F_{y_1} dy_1$ independent of the path in $\mathcal{H}'$ when the arguments of the function F_{y_1} are y , $p(y)$, $\lambda_\beta(y)$.

It should be noted that if a set $p_1(y)$, $\lambda_\beta(y)$ has the properties described above, then every set $k(y)p_1(y)$, $\lambda_\beta(y)$ for $k(y)$ a positive function with continuous derivatives for (y) in $\mathcal{H}'$ also has these properties on account of the homogeneity of the functions F and ϕ_β and the fact that the region $\mathcal{R}_1$ contains the point (y, ky') for k positive if it contains the point (y, y') . In particular, I^* is unchanged, and it will be shown later that the extremals of the field remain the same. Since the function ϕ appearing in the function ϕ_0 for problem C has the properties prescribed for $k(y)$, the slope functions of the field $\mathcal{H}'$ can be replaced by $p(y)/\phi[y, p(y)]$ and in this form they will satisfy the equation $\phi_0 = 0$. Furthermore, the corresponding cylindrical region $\bar{\mathcal{H}}'$ of (t, y) -space with slope functions and multipliers $p_1(y)/\phi[y, p(y)]$, $\lambda_0 = 0$, $\lambda_\beta(y)$ is a Mayer field for problem C. Whenever the problems B and C are being compared it will be assumed that the functions $p_1(y)$ for problem B have been replaced by $p_1(y)/\phi[y, p(y)]$ although, as noted above, the results for problem B will be independent of this transformation. Since the solutions of the equations $y_1' = p_1(y)/\phi[y, p(y)]$ are extremals of the field $\bar{\mathcal{H}}'$ with multipliers $\lambda_0 = 0$, $\lambda_\beta[y(t)]$, their projections in y -space are

extremals for problem B with multipliers $\lambda_\beta[y(t)]$. By the method of Section 4, the initial values for the functions y_1 can be expressed in terms of $n-1$ independent parameters. By replacing the parameter t by $\tau\phi(y, p)$ it is evident that every field $\mathcal{H}'$ for problem B is simply covered by an $(n-1)$ -parameter family of extremals satisfying the equations $y_1' = p_1(y)$. On an extremal arc E of a field $I^*(E) = I(E)$ since along such an arc the integrand of I^* reduces to F and $I(E) = \int_{t_1}^{t_2} F dt$.

Let $\bar{\mathcal{D}}$ be a region of points (y, y', λ_β) with (y, y') interior to the region $\mathcal{R}_1$ described in Section 2 and so related to a region $\mathcal{N}$ of points (y, q) that to each point (y, q) interior to $\mathcal{N}$ there corresponds a unique solution $(y, y', \lambda_0 = 0, \lambda_\beta)$ of the equations (3:5) for problem C with (y, y', λ_β) interior to $\bar{\mathcal{D}}$ and such that the determinant of the matrix R in (3:6) is different from zero. Let the region N for problem C, corresponding to the region N for problem A described in Section 6, be the region of points (t, y, q) such that (y, q) is in $\mathcal{N}$; similarly, let the region $\bar{\mathcal{D}}$ for problem C, corresponding to the region $\bar{\mathcal{D}}$ for problem A, be the region of points $(t, y, y', \lambda_0, \lambda_\beta)$ such that (y, y', λ_β) is interior to $\bar{\mathcal{D}}$. The equations (4:3) with $z_1 = q_1$ have as before single-valued solutions with continuous second derivatives and $\lambda_0 = H(y, q) = \text{constant}$ over the interior of N ; furthermore every solution $y_1(t)$, $z_1 = q_1(t)$ of equations (4:8) in the region N defines an extremal for problem C whose projection in the y -space is an extremal for problem B with $[y(t), q(t)]$ interior to the region $\mathcal{N}$. A set (y, q) interior to $\mathcal{N}$ will be called an admissible set.

A second definition of a Mayer field is a region $\mathcal{F}$ of y -space containing only interior points with which are associated functions $q_1(y)$ having continuous first partial derivatives in $\mathcal{F}$,

defining elements (y, q) which lie in the region $\mathcal{N}$ and satisfy the equation $H(y, q) = 0$ for every (y) in $\mathcal{F}$, and making the integral $I^* = \int q_1(y) dy_1$ independent of the path in $\mathcal{F}$.

If $\mathcal{F}$ is a field for problem B, then the corresponding cylindrical region of (ty) -space is a field $\bar{\mathcal{F}}$ for problem C with the same functions $q_1(y)$ and the same invariant integral since $H[y, q(y)] = 0$. The region N for problem C is the region of points (t, y, q) such that (y, q) is in $\mathcal{N}$ and t is arbitrary.

THEOREM 10:1. If the integral I^* is independent of the path in $\mathcal{F}$, the equations

$$\partial q_1 / \partial y_k = \partial q_k / \partial y_1 \quad (1, k = 1, \dots, n)$$

hold identically in the variables (y) .

This theorem follows at once from Theorem 6:1 for problem C.

THEOREM 10:2. Every solution $y_1(t)$ in the field $\mathcal{F}$ of the equations

$$(10:1) \quad dy_1/dt = H_{z_1}[y, q(y)]$$

is an extremal with $z_1(t) = q_1[y(t)]$. Through each point of the field there passes one and only one extremal.

This theorem follows from Theorem 6:2 for problem C. Every solution $y_1(t)$ in the field $\bar{\mathcal{F}}$ of the equations (10:1) with $z_1(t) = q_1[y(t)]$ satisfies the equations

$$(10:2) \quad dy_1/dt = H_{z_1}(y, z), \quad dz_1/dt = -H_{y_1}(y, z), \quad H(y, z) = 0,$$

which are the canonical equations of the extremals. The projections in the y -space of the arcs satisfying the equations (10:2) are called the extremals of the field $\mathcal{F}$. By the method of Section 4 the n parameters in the solutions of equations (10:1) can be

expressed in terms of $(n-1)$ independent parameters in such a way that through each point of $\mathcal{F}$ there will pass one and only one extremal of the field.

A field $\mathcal{F}$ determined by functions $q_1(y)$ is a Mayer field as defined by Bliss with $p_1(y) = H_{z_1}[y, q(y)]$, $\lambda_\beta = L_\beta[y, q(y)]$. A Mayer field defined by functions $p_1(y)$, $\lambda_\beta(y)$ is a field according to the second definition provided that the sets $[y, q(y)]$ defined by points (y) in $\mathcal{F}'$, where $q_1(y) = F_{y_1}[y, p(y), \lambda(y)]$, are interior to the region $\mathcal{N}$ and satisfy the equation $H(y, q) = 0$.

Let $\mathcal{F}$ be a field defined by functions $q_1(y)$. An n -space $W(y) = \text{constant}$ is called a transversal surface of the field if at each point (y) on it the equation

$$q_1(y)dy_1 = 0$$

is satisfied by every set of differentials dy_1 which satisfy the equation

$$W_{y_1}dy_1 = 0.$$

It is evident that a necessary and sufficient condition that an n -space $W(y) = \text{constant}$ be a transversal surface of a field $\mathcal{F}$ is that at each point on it in $\mathcal{F}$

$$W_{y_1} = kq_1(y)$$

where k is a proportionality factor.

THEOREM 10:3. In every field $\mathcal{F}$ the Hilbert Integral defines a function $W(y)$ having continuous second partial derivatives in $\mathcal{F}$, satisfying the Hamilton-Jacobi partial differential equation

$$(10:3) \quad H(y, W_y) = 0,$$

and such that the surfaces of the family $W(y) = \text{constant}$ are the transversal surfaces of the field. Conversely, let $W(y)$ be a function which has continuous second partial derivatives and defines only admissible elements $(y, q) = (y, W_y)$ in a region $\mathcal{H}$ of y -space. Then if W satisfies the Hamilton-Jacobi partial differential equation (10:3), the region $\mathcal{H}$ is a field with

$$q_1(y) = W_{y_1},$$

and the surfaces $W(y) = \text{constant}$ are the transversal surfaces of the field.

This theorem follows at once from Theorem 7:1 for problem C with the aid of the remarks preceding Theorem 10:1.

The following properties are easily established for a field $\mathcal{H}$. In a field, $I^*(C_{12}) = W(y)|_1^2$ for any two points 1 and 2. Consequently, for every arc C_{12} on a transversal surface, $I^*(C_{12})$ is identically zero. From this it follows that, if the extremals of the field do not lie on the transversal surfaces, then two transversal surfaces intercept arcs on the extremals of the field along which the integral $\int z_1 H_{z_1} dt$ has a constant value.

11. Construction of special fields and the relations between extremals and a complete integral of the Hamilton-Jacobi equation for problem C. In this section it is shown how to construct fields for problem C for which the functions q_1 are independent of t , and the special form which the theory developed in Section 8 assumes for this problem is given.

LEMMA 11:1. If $y_1(t)$, $z_1(t)$ is a solution of equations (4:8) for problem C and the equation $H(y, z) = \text{constant}$, then every set of the form $\eta_1 = cy_1'$, $\zeta_1 = cz_1'$, where c is a constant, is a solution of the canonical accessory equations (8:7) and the equation

$$(11:1) \quad H_{y_1} \gamma_1 + H_{z_1} \zeta_1 = 0.$$

This statement is readily verified by direct substitution.

THEOREM 11:1. If, in Theorem 8:2 for problem C, the conjugate system of solutions of equations (8:7) is of the form $y_1^i, z_1^i, \gamma_{1r}(t), \zeta_{1r}(t)$ ($r = 1, \dots, n-1$) and satisfy the equation (11:1), then by proper choice of the parameters the functions q_1 for each field are independent of t and the Hamilton function $H(y, q)$ associated with each field is a constant χ_0 . The invariant integral described in Theorem 8:3 has the form

$$(11:2) \quad W(t, y, \beta, \chi_0) = \int_{(y_0)}^{(y)} q_1(y, \beta, \chi_0) dy_1 - \chi_0 t = W(y, \beta, \chi_0) - \chi_0 t.$$

The theorem will be established by a method suggested by the proof of a theorem due to Bliss (V, 241). Let E' be a member for values $t_{10} \leq t \leq t_{20}$, a_{10} , b_{10} of a $2n$ -parameter family of extremals

$$(11:3) \quad y_1 = y_1(t-t_0, a, b), \quad z_1 = z_1(t-t_0, a, b)$$

where the parameters a_1, b_1 are the values of the functions y_1, z_1 , at $t = t_0$. By Theorem 4:3 the Hamilton function $H(y, z)$ has a constant value $H(a, b)$ along each extremal of the family (11:3). The functions y_1, y_{1t}, z_1, z_{1t} are assumed to have continuous first and second partial derivatives in a neighborhood of the values (t, a, b) on E' . The parameter t_0 appears only in the difference $t - t_0$ since the right members of equations (4:8) for problem C do not contain t explicitly. It is proposed to construct a Mayer family of fields containing E' for which the functions q_1 associated with each field are independent of t . To do this, a function $U(\alpha, \beta)$ is defined by the equation

$$(11:4) \quad U(\alpha, \beta) = \frac{1}{2} \zeta_{1r}(t_0) \gamma_{1j}(t_0) \alpha_r \alpha_j + \alpha_r \beta_r \quad (r=1, \dots, n-1)$$

and the parameters a_1, b_1 are determined as functions of α, β, χ_0 by means of the equations

$$(11:5) \quad a_1(\alpha) = a_{10} + \gamma_{1r}(t_0) \alpha_r \quad (1=1, \dots, n) \\ (j, r=1, \dots, n-1)$$

$$(11:6) \quad b_1 \gamma_{1j}(t_0) = U_{\alpha_j} = \zeta_{1r}(t_0) \gamma_{1j}(t_0) \alpha_r + \beta_j, \quad H[a(\alpha), b] = \chi_0.$$

These equations are satisfied at the special values

$(\alpha_j, \beta_j, \chi_0, a_1, b_1) = [0, b_{10} \gamma_{1j}(t_0), \chi_0(E'), a_{10}, b_{10}]$. For these special values the functional determinant of the left members of equations (11:6) with respect to the variables b_1 is the determinant $|\gamma_{1j}(t_0) \quad y_1'|$ which is different from zero along E' by hypothesis. Hence, equations (11:6) have solutions $b_1(\alpha, \beta, \chi_0)$ in a neighborhood of values $[\alpha_{r0}, \beta_{r0}, \chi_0(E')] = [0, b_{10} \gamma_{1r}(t_0), \chi_0(E')]$ which are such that $b_1[\alpha_0, \beta_0, \chi_0(E')] = b_{10}$.

If the functions $a(\alpha), b(\alpha, \beta, \chi_0)$ are substituted in equations (11:6) and the resulting identities are differentiated for α_r, β_r , it is seen that

$$(11:7) \quad b_{1\alpha_r} \gamma_{1j}(t_0) = \zeta_{1r}(t_0) \gamma_{1j}(t_0), \quad b_{1\alpha_r} H_{z_1} = -H_{y_1} \gamma_{1r}(t_0)$$

$$(11:8) \quad b_{1\beta_r} \gamma_{1j}(t_0) = \delta_{rj}, \quad b_{1\beta_r} H_{z_1} = 0, \quad b_{1\chi_0} \gamma_{1j}(t_0) = 0,$$

$$b_{1\chi_0} H_{z_1} = 1,$$

where $\delta_{rr} = 1$ and $\delta_{rj} = 0$ if $r \neq j$. Since along E' the solutions $y_1', z_1', \gamma_{1j}(t), \zeta_{1j}(t)$ of equations (8:7) and (11:1) are assumed to form a conjugate system, it follows with the aid of equations (11:7) that along E'

$$[b_{1\alpha_r} - \zeta_{1r}(t_0)] \gamma_{1j}(t_0) = 0, \quad [b_{1\alpha_r} - \zeta_{1r}(t_0)] y_1' = 0.$$

Since the determinant $|\gamma_{1j}(t_0) \quad y_1^1|$ is different from zero on E^1 , the equations $b_{1\alpha_r} = \zeta_{1r}(t_0)$ must hold identically on E^1 . Furthermore, from equations (11:8) with $H_{z_1} = y_1^1$, it is seen that at $t = t_0$ on E^1

$$(11:9) \quad \begin{vmatrix} b_{1\beta_r} \\ b_{1\chi_0} \end{vmatrix} \cdot |\gamma_{1j}(t_0) \quad y_1^1| = \begin{vmatrix} \delta_{rj} & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

and hence that the first determinant is different from zero.

The $2n$ -parameter family of extremals

$$(11:10) \quad y_1 = Y_1(t-t_0, \alpha_r, (\beta_r, \chi_0), \quad z_1 = Z_1(t-t_0, \alpha_r, (\beta_r, \chi_0) \\ (r = 1, \dots, n-1)$$

obtained from equations (11:3) by substituting $a_1(\alpha), b_1(\alpha, \beta, \chi_0)$ for a_1 and b_1 contains the extremal E^1 for $(t_0, \alpha_r, (\beta_r, \chi_0) = [t_{00}, 0, b_{10} \gamma_{1r}(t_0), \chi_0(E^1)]$ and can be shown to have the properties stated in the theorem.

By the method used in the proof of Theorem 8:2, it is readily established that at $t = t_0$ the variations of the family (11:10) with respect to the parameters are

$$Y_{1t_0} = -y_1^1, \quad Y_{1\alpha_r} = \gamma_{1r}(t_0), \quad Z_{1t_0} = -z_1^1, \quad Z_{1\alpha_r} = \zeta_{1r}(t_0), \\ (11:11)$$

$$Y_{1\chi_0} = 0, \quad Y_{1\beta_r} = 0, \quad Z_{1\chi_0} = b_{1\chi_0}, \quad Z_{1\beta_r} = b_{1\beta_r},$$

and hence, that on E^1

$$(11:12) \quad Y_{1t_0} = -y_1^1, \quad Y_{1\alpha_r} = \gamma_{1r}(t).$$

From the relations (11:11) and (11:9) it follows that the second determinant in (8:2) is different from zero at t_0 and consequently different from zero along E^1 . From the equations (11:12) it is seen that along E^1 , $|Y_{1t_0} \quad Y_{1\alpha_r}| = |-y_1^1 \quad \gamma_{1r}(t)|$, and the

latter determinant is different from zero. It follows that the first n equations in (11:10) can be solved for t_0, α_r as functions of t, β, λ_0 . If these solutions $t_0(t, \beta, \lambda_0), \alpha_r(t, \beta, \lambda_0)$ are substituted in the first n equations in (11:10), and the resulting identities are differentiated for t , one obtains the relation

$$0 = Y_1^i (1 - t_0^i) + Y_1 \alpha_r \alpha_r^i.$$

Since the determinant $|Y_1^i Y_1 \alpha_r|$ is different from zero for $(t, t_0, \alpha, \beta, \lambda_0)$ in a neighborhood $\mathcal{B}$ of the values on E^1 , it follows that $t_0^i = 1, \alpha_k^i = 0$. Hence, the functions $t_0(t, \beta, \lambda_0), \alpha_r(t, \beta, \lambda_0)$ are of the form

$$(11:13) \quad t_0 = t - t(y, \beta, \lambda_0), \quad \alpha_r = \alpha_r(y, \beta, \lambda_0)$$

for (β, λ_0) in $\mathcal{B}$ and (t, y) in a neighborhood $\mathcal{F}$ of the values on E^1 . With the aid of equations (11:13), functions $q_1(t, y, \beta, \lambda_0)$ are defined by the equations

$$(11:14) \quad q_1 = Z_1[t(y, \beta, \lambda_0), \alpha(y, \beta, \lambda_0), \beta, \lambda_0]$$

obtained by substituting the relations (11:13) in the last n equations in (11:10). It is clear that the functions q_1 are independent of t . With the aid of the last equation in (11:6) it follows that

$$(11:15) \quad H(y, q(y, \beta, \lambda_0)) = \lambda_0$$

is identically satisfied for each fixed set (β, λ_0) in $\mathcal{B}$ and (y) in $\mathcal{F}$. On the hyperplane $y_1 = a_{10} + \gamma_{1r}(t_0) \alpha_r$, on account of the relations (11:5), (11:6), (11:11) and (11:15) the integral I^* is independent of the path since

$$I^* = \int_C b_1 \gamma_{1r}(t_0) d\alpha_r - H[a(\alpha), b(\alpha, \beta, \lambda_0)] dt = \int_C dU - \lambda_0 dt.$$

The integral I^* is therefore independent of the path in the whole neighborhood $\mathcal{H}^1$ (IV, 106) and for each fixed set (β, λ_0) in $\mathcal{B}$ the region $\mathcal{H}^1$ is a field with the functions $q_1(y, \beta, \lambda_0)$ defined in (11:14). The value of the integral I^* taken from a fixed point (t_0, y_0) in $\mathcal{H}^1$ to a variable point (t, y) has the form given in (11:2) on account of the definitions (11:14) and the identity (11:15).

THEOREM 11:2. In each field $\mathcal{H}^1$ described in Theorem 11:1, the function $\bar{W}$ in (11:2) with an additive constant is a complete integral of the Hamilton-Jacobi differential equation

$$\bar{W}_t + H(y, \bar{W}_y) = 0$$

and has the determinant $|\bar{W}_{y_1} \partial_r \bar{W}_{y_1} \lambda_0|$ ($r = 1, \dots, n-1$) different from zero.

This theorem follows from Theorem 11:1 and the first part of Theorem 7:1.

THEOREM 11:3. If $\bar{W}(t, y_1, \dots, y_n, \beta_1, \dots, \beta_{n-1}, \lambda_0)$ has the properties described in Theorem 8:4 for problem C and is of the form $\bar{W}(t, y, \beta, \lambda_0) = W(y, \beta, \lambda_0) - \lambda_0 t$, then the equations

$$W_{\beta_r}(y, \beta, \lambda_0) = W_{\beta_r}[a(\alpha), \beta, \lambda_0],$$

$$W_{\lambda_0}(y, \beta, \lambda_0) = W_{\lambda_0}[a(\alpha), \beta, \lambda_0] + t - t_0 \quad (r = 1, \dots, n-1)$$

define a $2n$ -parameter family of extremals of the form (11:10) with properties similar to those described in Theorem 4:1 and which satisfy the equation

$$(11:16) \quad H(y, W_y) = \lambda_0$$

identically in t, y, β, λ_0 , the functions $z_1(t-t_0, \alpha, \beta, \lambda_0)$ associated

with these extremals being defined by the equations

$$z_1 = W_{y_1}(y, \beta, \lambda_0).$$

The functions $a_1(\alpha_1, \dots, \alpha_{n-1})$ are required to have continuous second partial derivatives for (α) in a region $\mathcal{A}$ and the determinant

$$(11:17) \quad \left| H_{z_1} \quad a_1 \alpha_r \right| \quad (i=1, \dots, n; r=1, \dots, n-1)$$

different from zero, where the arguments in H_{z_1} are $a_1(\alpha)$, $W_{y_1}[a(\alpha), \beta, \lambda_0]$.

This theorem follows from Theorem 8:4 with $a_1 = a_1(\alpha)$ and t_0 variable. With the aid of equation (11:16) and the fact that the determinant (11:17) is different from zero, it can be proved, using a method analogous to that used in the proof of Theorem 8:4, that the functional determinant of the variables (y, z) with respect to the parameters $(t_0, \alpha, \beta, \lambda_0)$ is different from zero.

THEOREM 11:4. If the last s of the parameters (β) do not appear in the functions $Y_1(t - t_0, \alpha, \beta, \lambda_0)$ in (11:10) of Theorem 11:1, where s is one of the numbers $1, \dots, n-1$, then the function $\bar{W}(t, y, \beta, \lambda_0) = W(y, \beta, \lambda_0) - \lambda_0 t$ described in Theorem 11:2 satisfies the additional equations

$$(11:18) \quad H_{z_1 z_k} \bar{W}_{y_k} \beta_\rho = H_{z_1 z_k} W_{y_k} \beta_\rho = 0 \quad (\rho = n-s, \dots, n-1).$$

Similarly, if the parameter λ_0 does not appear in the functions $Y_1(t - t_0, \alpha, \beta, \lambda_0)$, then the function $\bar{W}$ satisfies the equations

$$(11:19) \quad H_{z_1 z_k} \bar{W}_{y_k} \lambda_0 = H_{z_1 z_k} W_{y_k} \lambda_0 = 0.$$

If the function $\bar{W}(t, y, \beta, \lambda_0) = W(y, \beta, \lambda_0) - \lambda_0 t$ described in

Theorem 11:3 has continuous third derivatives and satisfies the equations (11:18), then s of the parameters (β) do not appear in the functions $Y_1(t-t_0, \alpha, \beta, \lambda_0)$ in (11:10) defining the extremals. Similarly, if the function $\bar{W}$ satisfies equation (11:19) then the parameter λ_0 does not appear in the functions Y_1 in (11:10).

This theorem follows from Theorem 8:5 for problem C with $a_1 = a_1(\alpha)$ and t replaced by $t-t_0$.

THEOREM 11:5. Let $W(y_1, \dots, y_n, \beta_1, \dots, \beta_{n-1})$ be a solution of the Hamilton-Jacobi equation $W_t + H(y, W_y) = 0$ for problem C, having continuous first and second derivatives for values (y) in $\mathcal{F}$ and (β) in $\mathcal{B}$ and such that the matrix $|| W_{y_1 \beta_r} ||$ ($r=1, \dots, n-1$) has rank $n-1$. There exists a solution of the Hamilton-Jacobi equation for problem C of the form

$$\bar{W}(t, y, \beta, \lambda_0) = W(y, \beta, \lambda_0) - \lambda_0 t$$

having continuous first and second derivatives for all values of (t, y, β, λ_0) with (y) in $\mathcal{F}$, (β) in $\mathcal{B}$ and λ_0 near $\lambda_0 = 0$, at least if the domains $\mathcal{F}$ and $\mathcal{B}$ are taken sufficiently small. Moreover, the matrix $|| W_{y_1 \beta_r} \quad W_{y_1 \lambda_0} ||$ has rank n in this domain and $\bar{W}(t, y, \beta, 0) = W(y, \beta)$.

To prove this let $a_1(\alpha_1, \dots, \alpha_{n-1})$ be functions having continuous first and second derivatives and such that the determinant

$$\begin{vmatrix} H_{z_1}(a(\alpha), W_y[a(\alpha), \beta]) & a_1 \alpha_r \end{vmatrix}$$

is different from zero and the values a_1 are in $\mathcal{F}$. Such functions always exist if the domains $\mathcal{F}$ and $\mathcal{B}$ are sufficiently small. Let $b_1 = b_1(\alpha, \beta, \lambda_0)$ be solutions of the equations

$$H[a(\alpha), b] = \chi_0, \quad b_1 a_1 \alpha_r = W_{y_1}[a(\alpha), \beta] a_1 \alpha_r.$$

The function $\bar{W}(t, y, \beta, \chi_0)$ may be determined as follows. Let

$$y_1(t - t_0, a, b), \quad z_1(t - t_0, a, b)$$

be a $2n$ -parameter family of extremals for problem C containing the extremals of the field defined by the functions $q_1(y, \beta) = W_{y_1}(y, \beta)$ for values $b_1 = W_{y_1}[a(\alpha), \beta]$, the values a_1, b_1 being the values of the functions $y_1(t - t_0, a, b), z_1(t - t_0, a, b)$ at $t = t_0$. As in the proof of Theorem 11:1, it is seen that the equations

$$y_1 = y_1[t - t_0, a(\alpha), b(\alpha, \beta, \chi_0)]$$

have solutions of the form $t_0 = t - t(y, \beta, \chi_0), \alpha_r = \alpha_r(y, \beta, \chi_0)$ and that the function

$$\bar{W}(t, y, \beta, \chi_0) = \int_{(t_0, y_0)}^{(t, y)} q_1 dy_1 - H(y, q) dt,$$

where $q_1(y, \beta, \chi_0)$ are the values of the functions $z_1[t - t_0, a(\alpha), \beta, \chi_0]$ when $t_0 = t - t(y, \beta, \chi_0), \alpha_r = \alpha_r(y, \beta, \chi_0)$, has the properties described in the theorem.

12. Construction of fields and the relations between extremals and a complete integral of the Hamilton-Jacobi equation for problem B. An analogue of Theorem 8:2 for problem A is given in the following theorem.

THEOREM 12:1. Let E be a non-singular extremal for problem B for which there exists a conjugate system $Y_1^i, Z_1^i, \gamma_{ir}, \delta_{ir}$ ($r = 1, \dots, n-1$) of solutions of the canonical accessory equations

$$\eta'_1 = H_{z_1 y_k} \eta_k + H_{z_1 z_k} \zeta_k, \quad \zeta'_1 = -H_{y_1 y_k} \eta_k - H_{y_1 z_k} \zeta_k$$

and the equation

$$H_{y_1} \eta_1 + H_{z_1} \zeta_1 = 0, \quad (1, k = 1, \dots, n)$$

with the determinant $|Y'_1 \quad \eta_{1r}(t)|$ different from zero along E.
Then E is a member of a $(2n-2)$ -parameter family of extremals

$$Y_1 = Y_1(t, \alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}),$$

(12:1)

$$Z_1 = Z_1(t, \alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1})$$

which are such that the matrices

$$\left\| \begin{array}{cc} Y'_1 & Y_1 \alpha_r \end{array} \right\|, \quad \left\| \begin{array}{ccc} Y'_1 & Y_1 \alpha_r & Y_1 \beta_r \\ Z'_1 & Z_1 \alpha_r & Z_1 \beta_r \end{array} \right\| \quad (r=1, \dots, n-1)$$

have ranks n and 2n - 1 respectively for (t, α, β) in a neighborhood of the values on E. Furthermore, for fixed values (β) in a region $\mathcal{B}$, the family (12:1) forms a Mayer field over a neighborhood $\mathcal{F}$ of points (Y) on E, and the functions $q_1(Y, \beta)$ associated with each field have the matrix

$$\left\| q_1 \beta_r \right\|$$

of rank n - 1. The functions $q_1(Y, \beta)$ are defined by the equations

$$(12:2) \quad q_1(Y, \beta) = Z_1[t(Y, \beta), \alpha(Y, \beta), \beta],$$

and the functions $t(Y, \beta), \alpha(Y, \beta)$ are found by solving the first n equations in (12:1).

This theorem follows from Theorem 11:1 with $\chi_0 = 0$. The field $\mathcal{F}$ for fixed values (β) in $\mathcal{B}$ is the projection in the y-space

of the field $\mathcal{F}'$ described in Theorem 11:1, and the extremals (12:1) are the projections in the y -space of those extremals in (11:10) for which $\chi_0 = 0$ and the parameter $t - t_0$ has been replaced by a new parameter t .

THEOREM 12:2. Let E be a non-singular extremal for problem B for which the hypotheses of Theorem 12:1 are satisfied. Then there is a neighborhood of E in y -space in which, with an arbitrary additive constant, the function

$$W(Y, \beta) = \int_{(Y_0)}^{(Y)} q_1(Y, \beta) dy_1$$

for q_1 defined by equations (12:2) is a complete integral of the Hamilton-Jacobi partial differential equation $H(Y, W_Y) = 0$ and has the matrix $|| W_{Y_1} \beta_r ||$ ($r = 1, \dots, n-1$) of rank $n-1$.

This theorem follows from Theorem 12:1 for the case $\chi_0 = 0$ and $t - t_0$ replaced by a new parameter t .

THEOREM 12:3. If $W(Y_1, \dots, Y_n, \beta_1, \dots, \beta_{n-1})$ has continuous second partial derivatives with the matrix

$$(12:3) \quad \left\| \left\| W_{Y_1} \beta_r \right\| \right\| \quad (i = 1, \dots, n; r = 1, \dots, n-1)$$

of rank $n-1$ for all points (Y, β) satisfying conditions of the form (Y) in a region $\mathcal{F}$ in y -space (β) in a region $\mathcal{B}$, and the sets $(Y, q) = (Y, W_Y)$ corresponding to these arguments admissible, and if furthermore W satisfies the Hamilton-Jacobi equation

$$(12:4) \quad H(Y, W_Y) = 0$$

identically in (Y, β) , then the equations

$$(12:5) \quad W_{\beta_r}(Y, \beta) = W_{\beta_r}[a(\alpha), \beta]$$

define a $(2n-2)$ -parameter family of extremals

$$(12:6) \quad Y_1 = Y_1[t, a(\alpha), \beta], \quad Z_1 = Z_1[t, a(\alpha), \beta]$$

with properties similar to those described in Theorem 4:6, the functions Z_1 associated with these extremals being defined by the equations

$$Z_1 = W_{Y_1}.$$

The functions $a_1(\alpha_1, \dots, \alpha_{n-1})$ are required to have continuous second partial derivatives for (α) in a region $\mathcal{A}$ and the determinant

$$\begin{vmatrix} H_{Z_1} & a_{1\alpha_r} \end{vmatrix} \quad (r = 1, \dots, n-1)$$

different from zero, where the arguments in H_{Z_1} are $a_1(\alpha)$, $W_{Y_1}[a(\alpha), \beta]$.

This theorem follows at once from Theorem 11:3 with the aid of Theorem 11:5. The extremals (12:6) are the projections in the y -space of these extremals for problem C for which $\lambda_0 = 0$ and $t - t_0$ is replaced by a new parameter t .

THEOREM 12:4. If the last s of the parameters (β) do not appear in the functions $Y_1(t, \alpha, \beta)$ in (12:1) of Theorem 12:1, where s is one of the numbers $1, \dots, n-1$, then the function $W(Y, \beta)$ described in Theorem 12:2 satisfies the additional equations

$$(12:7) \quad H_{Z_1 Z_k} W_{Y_k} \beta_\rho = 0 \quad (\rho = n-s, \dots, n-1).$$

If the function $W(Y, \beta)$ described in Theorem 12:3 has continuous third derivatives and also satisfies equations (12:7), then the last s of the parameters (β) do not appear in the functions $Y_1[t, a(\alpha), \beta]$ in (12:6) defining the extremals.

The first statement follows at once from the first part of Theorem 11:4 by setting $\lambda_0 = 0$ and replacing the parameter $t - t_0$ by a new parameter t in the right members of the equations (11:10). With the aid of Theorem 11:5, the final statement in the theorem follows in a similar way from the last part of Theorem 11:4.

13. Fields for the problem of Mayer in parametric form.

The theorems in this section are analogous to those for the non-parametric problem in Section 9.

THEOREM 13:1. The integral I^* has the value zero along every extremal of a field $\mathcal{F}$. Furthermore, every extremal of a field lies on one of its transversal surfaces $W(y) = \text{constant}$ and the values W_{y_1} are not all zero.

The first statement follows from the homogeneity of the function H given in (4:20) and the fact that $H[y, q(y)]$ is identically zero by the definition of a field for problem B. The final statement in the theorem is a consequence of the first part of Theorem 10:3 and the fact that W_{y_1} are the values of the functions z_1 belonging to the extremals of the field.

THEOREM 13:2. Let E' be an extremal arc for the problem of Mayer in non-parametric form corresponding to an extremal arc E for the problem of Mayer in parametric form and having the properties described in Theorem 11:1. Then by proper choice of the parameters the family of extremals in (11:10) will satisfy the homogeneity relations (4:19) and the functions $q_1(y, \beta, \lambda_0)$ described in (11:14) will satisfy the conditions

$$(13:1) \quad q_1(y, k\beta, k\lambda_0) = kq_1(y, \beta, \lambda_0) \quad (k \neq 0).$$

The proof is like that of Theorem 11:1 if the function $U(\alpha, \beta)$ in equation (11:4) is replaced by a suitable function of similar type that is homogeneous of order one in the parameters $\beta_1, \dots, \beta_{n-1}$. It should be noted that not all of the expressions $b_{10}\eta_{1r}(t_0)$ can be zero. This is a consequence of the condition (3:7) which reduces to the form $z_1 y_1' = \lambda_0 = 0$, the relation $z_1 \eta_{1r}(t) = \text{constant}$ which follows from the equations $\bar{\Phi}_\alpha = 0$ in (2:1) with the aid of the Euler-Lagrange equations (3:9)(III, 307), and the hypothesis that the determinant $|y_1' \eta_{1r}(t)|$ is different from zero along E' . It may be supposed, therefore, that $b_{10}\eta_{10}(t_0)$ is not zero. Then the function

$$U(\alpha, \beta) = \frac{1}{2} \sum_{ir} \eta_{1r}(t_0) \eta_{1j}(t_0) \alpha_r \alpha_j \beta_1 + b_{10} \eta_{11}(t_0) \alpha_1 \beta_1 + \alpha_s \beta_s$$

$$(r, j=1, \dots, n-1; s=2, \dots, n-1),$$

has the required properties. The condition (13:1) can be established by the method used to establish the condition (9:1) in the proof of Theorem 9:2.

THEOREM 13:3. Let E be a non-singular extremal arc for the problem of Mayer in parametric form having the properties described in Theorem 12:1. Then by proper choice of the parameters the family of extremals in (12:1) will satisfy the homogeneity relations (4:21) and the functions $q_1(Y, \beta)$ defined in (12:2) will satisfy the conditions

$$q_1(Y, k\beta) = kq_1(Y, \beta), \quad (k \neq 0).$$

This theorem follows at once from Theorem 13:2 and Theorem 12:1.

THEOREM 13:4. If E is a non-singular extremal arc for the problem of Mayer satisfying the hypotheses of Theorem 13:3,

then the function W described in Theorem 12:2, without the additive constant, can be put into the form $\beta_1 U(Y_1, \dots, Y_n, c_1, \dots, c_{n-2})$ by proper choice of the parameters and the number s described in the first part of Theorem 12:4 is at least equal to one. Conversely, if a function $U(Y_1, \dots, Y_n, c_1, \dots, c_{n-2})$ has continuous second partial derivatives with the matrix

$$\left\| \begin{array}{c} U_{Y_1} \\ U_{Y_1 c_j} \end{array} \right\| \quad (i=1, \dots, n; j=1, \dots, n-2)$$

of rank $n-1$ for all points (Y, c) satisfying conditions of the form (Y) in a region $\mathcal{F}$, (c) in a region $\mathcal{C}$, and the sets $(Y, c) = (Y, U_Y)$ corresponding to these arguments admissible, and if U satisfies the Hamilton-Jacobi equation (12:4) identically in (Y, c) , then $W = \beta_1 U(Y_1, \dots, Y_n, c_1, \dots, c_{n-2})$ for $\beta_1 \neq 0$ is a function possessing the properties of the function W described in Theorem 12:3. If U also has continuous third derivatives, then the number s described in the last part of Theorem 12:4 is at least equal to 1.

The first part of the theorem is an immediate consequence of Theorem 13:3. The final statement of the theorem follows from the homogeneity of the function H given in (4:20) and its easily deduced consequence $z_1 H_{z_1} z_k = 0$.

THEOREM 13:5. Each transversal surface $W(y) = c$ of a field $\mathcal{F}$ determined by functions $q_1(y)$ contains an $(n-2)$ -parameter family of extremals of the field defined by functions of the form

$$(13:3) \quad y_1(t, \alpha_1, \dots, \alpha_{n-2}, c), \quad z_1(t, \alpha_1, \dots, \alpha_{n-2}, c).$$

At each point on the transversal surface, the equations

$$(13:4) \quad q_1(y) y_1 \alpha_k = 0 \quad (i=1, \dots, n; k=1, \dots, n-2)$$

are identically satisfied by the functions $y_1(t, d, c)$ in (13:3).

By Theorem 10:3 the transversal surfaces of the field $\mathcal{F}$ are defined by the equations $W(y) = \text{constant}$, where

$$W(y) = \int_{y_0}^y q_1(y) dy_1,$$

in which y_0 is a fixed point of $\mathcal{F}$ and $q_1(y)$ are the functions defining the field. The extremals of the field are given by functions

$$y_1(t, \alpha_1, \dots, \alpha_{n-1}), \quad z_1(t, \alpha_1, \dots, \alpha_{n-1}),$$

having continuous first and second derivatives such that the determinant $|y_1' \ y_1 \alpha_r|$ ($r = 1, \dots, n-1$) is different from zero in $\mathcal{F}$. In view of Theorem 13:1, we have the identity

$$W[y(t, \alpha)] = W[y(t_0, \alpha)]$$

where t_0 is a fixed value of t . Moreover the values

$$(13:5) \quad W_{y_1}[y(t, \alpha)] y_1 \alpha_r$$

are not all zero at $t = t_0$. To prove this, recall that the relations

$$z_1 = W_{y_1}(y), \quad z_1 y_1' = z_1 H_{z_1}(y, z) = H = 0,$$

hold with $y_1 = y_1(t, \alpha)$, $z_1 = z_1(t, \alpha)$. Since the z 's are not all zero it follows that if the values (13:5) were all zero at $t = t_0$, the determinant $|y_1' \ y_1 \alpha_r|$ would vanish at $t = t_0$ which is not the case.

By means of the equations $W[y(t_0, \alpha)] = c$ it is seen that the parameters $\alpha_1, \dots, \alpha_{n-1}$ can be replaced by a new set of $n-1$ parameters one of which is the constant c . We may accordingly

suppose that $\alpha_{n-1} = c$. This proves the first part of the theorem. The second part is obtained by differentiating the equation $W[y(t, \alpha_1, \dots, \alpha_{n-2}, c)] = c$ for α'_k ($k = 1, \dots, n-2$) and replacing W_{y_1} by $q_1(y)$.

14. Relation of Kneser field theory for the problem of Mayer in non-parametric form to the field theory developed in this paper. Let E be a non-singular extremal which is a member for values $t_1 \leq t \leq t_2$, $a_{10}, \dots, a_{n-1,0}$ of an $(n-1)$ -parameter family of extremals

$$(14:1) \quad y_1 = y_1(t, a_1, \dots, a_{n-1}), \quad z_1 = z_1(t, a_1, \dots, a_{n-1})$$

which are such that along each extremal the matrix

$$(14:2) \quad ||y_{1a_k}|| \quad (i = 1, \dots, n; k = 1, \dots, n-1)$$

has rank $n-1$ and the equations

$$(14:3) \quad z_1 y_{1a_k} = F_{y_1} y_{1a_k} = 0$$

are identically satisfied. Then the family of extremals (14:1) defines a field in the sense of Kneser (IX, 43).

Since along each extremal the function F for the problem of Mayer is identically zero, it follows readily with the aid of the Euler-Lagrange equations (3:5) that $F_{y_1} y_{1a_k} = \text{constant}$ along each extremal (III, 307). Hence, if the constant is zero at the intersection of the extremal arcs in (14:1) with a sub-space defined by the equation $t = t(a_1, \dots, a_{n-1})$, then the condition (14:3) is satisfied along each extremal of the family.

THEOREM 14:1. Along each extremal of a Kneser field, the equation

$$(F - y_1' F_{y_1'}) dt + F_{y_1'} dy_1 = 0$$

holds identically where $dy_1 = y_1' dt + y_1 a_k da_k$. Furthermore, the extremals of a Kneser field lie on a surface which satisfies the Hamilton-Jacobi partial differential equation $W_t + H(t, y, W_y) = 0$.

The first statement is an immediate consequence of the definition of a Kneser field and the fact that the function F is identically zero along each extremal. The analogue of this condition for a Mayer field in the sense of Bliss is the vanishing of the integral I^* along each extremal of a field. The second statement in the theorem is readily established by the method Kneser used for the parametric problem (IX, 44). The surface is obtained by eliminating the parameters from the first n equations in (14:1). This last result also follows readily from Theorems 9:1, 9:5 and the following theorem.

THEOREM 14:2. A Mayer field in the sense of Bliss consists of a one-parameter family of Kneser fields.

This theorem is an immediate consequence of Theorem 9:5. The condition (9:3) which must hold on each transversal surface of a Mayer field in the sense of Bliss is the condition (14:3) for a Kneser field.

THEOREM 14:3. Every Kneser field can be imbedded in a one-parameter family of Kneser fields forming a Mayer field in the sense of Bliss, at least if the Kneser field is sufficiently small.

Let E be a non-singular extremal arc imbedded for values $t_{10} \leq t \leq t_{20}$, α_{r0} in an $(n-1)$ -parameter family of extremals

$$y_1 = y_1(t, \alpha_1, \dots, \alpha_{n-1}), \quad z_1 = z_1(t, \alpha_1, \dots, \alpha_{n-1})$$

which forms a Kneser field $\mathcal{F}'$ for (α) in a region a . Set

$$(14:4) \quad y_1(t_0, \alpha) = A_1(\alpha), \quad z_1(t_0, \alpha) = B_1(\alpha)$$

and denote their values at $\alpha_r = \alpha_{r_0}$ by a_{10} , b_{10} , respectively. From the definition of Kneser fields it follows that

$$(14:5) \quad B_1 A_1 \alpha_r = 0 \quad (r=1, \dots, n-1),$$

and the matrix $||A_1 \alpha_r||$ has rank $n-1$. As was seen in Section 4 the extremal E can be imbedded for values $t_{10} \leq t \leq t_{20}$, a_{10} , b_{10} in a $2n$ -parameter family of extremals

$$(14:6) \quad y_1 = y_1(t, a, b), \quad z_1 = z_1(t, a, b)$$

of the type described in Theorem 4:1, where a_1 , b_1 are the values of y_1 , z_1 at $t = t_0$ and the parameters b_1 do not vanish simultaneously.

Let $2n$ functions $a_1(\alpha, c)$, $b_1(\alpha, c)$ be defined by the equations

$$(14:7) \quad a_1(\alpha, c) = A_1(\alpha) + cB_1(\alpha),$$

$$(14:8) \quad b_1(A_1 \alpha_r + cB_1 \alpha_r) = 0, \quad b_1 B_1 = B_1(\alpha) B_1(\alpha),$$

where $A_1(\alpha)$, $B_1(\alpha)$ are the functions in (14:4). With the aid of conditions (14:5), it is seen that the equations (14:7) and (14:8) are satisfied at the special values $(\alpha, c, a, b) = [\alpha_{r_0}, 0, a_{10}, b_{10} = B_1(\alpha_{r_0})]$. The functional determinant of the left members of equations (14:8) with respect to the parameters b_1 is the determinant $|A_1 \alpha_r + cB_1 \alpha_r \quad B_1|$. For the special values given above, this determinant is different from zero. For, if it were equal to zero, there would exist constants d_r , d not all zero such that $d_r A_1 \alpha_r + dB_1 = 0$. Multiplying each of these equations by B_1 and adding the equations, one finds that $d = 0$ on account

of the conditions (14:5) and the fact that $B_1 B_1 \neq 0$. It follows that $d_r A_1 \alpha_r = 0$ for constants d_r not all zero. This contradicts the hypothesis that the matrix $|| A_1 \alpha_r ||$ has rank $n-1$. Hence, for (α, c) in a neighborhood of $(\alpha_{r_0}, 0)$, the equations (14:8) have solutions $b_1 = b_1(\alpha, c)$ which are such that $b_1(\alpha_{r_0}, 0) = b_{10}$ and $b_1(\alpha, 0) = B_1(\alpha)$. The n -parameter family of extremals

$$y_1 = y_1[t, A(\alpha) + cB(\alpha), b(\alpha, c)] = Y_1(t, \alpha, c),$$

(14:9)

$$z_1 = z_1[t, A(\alpha) + cB(\alpha), b(\alpha, c)] = Z_1(t, \alpha, c),$$

obtained from the family (14:6) by substituting $a_1(\alpha, c)$, $b_1(\alpha, c)$ for a_1 , b_1 , respectively, contains E for $\alpha_r = \alpha_{r_0}$, $c = 0$, and will be shown to have the properties required for the theorem.

It will first be shown that the determinant $|Y_1 \alpha_r \quad Y_{1c}|$ is different from zero along E . If this determinant were equal to zero at a point on E , there would exist constants d_r , d not all zero such that $d_r Y_1 \alpha_r + d Y_{1c} = 0$. Multiplying each of these equations by Z_1 and adding, one obtains the equation

$$d_r Z_1 Y_1 \alpha_r + d Z_1 Y_{1c} = 0.$$

The expressions $Z_1 Y_1 \alpha_r$, $Z_1 Y_{1c}$ are constant along each extremal (III, 307) and since the first set vanishes at $t = t_0$ on account of condition (14:5), while the second set does not on account of the last group of equations in (14:8), it follows that $d = 0$. Hence, for constants d_r not all zero, $d_r Y_1 \alpha_r = 0$, which contradicts the hypothesis that the matrix (14:2) has rank $n-1$.

Furthermore, for values of α_r and each fixed value of c in a neighborhood of the values $(\alpha_{r_0}, 0)$, the family (14:9) defines a Kneser field. For, the matrix $|| Y_1 \alpha_r ||$ has rank $n-1$ and the expressions $Z_1 Y_1 \alpha_r$ are identically zero on account of the first

group of equations in (14:8).

Finally, the family (14:9) defines a Mayer field in a neighborhood $\mathcal{F}$ of the values (t, y) on E , with functions $q_1(t, y)$ defined by the equations

$$q_1(t, y) = Z_1[t, \alpha(t, y), c(t, y)],$$

where $\alpha_r(t, y)$, $c(t, y)$ are found by solving the first n equations in (14:9). To establish this statement it remains to be shown that the integral I^* is independent of the path in $\mathcal{F}$. On the hyperplane $t = t_0$, it is seen with the aid of the equations (14:7) and (14:8) that

$$\begin{aligned} \int_C q_1(t, y) dy_1 &= \int_C b_1(\alpha, c) [A_1 \alpha_r + c B_1 \alpha_r] d\alpha_r + b_1(\alpha, c) B_1 dc \\ &= B_1(\alpha) B_1(\alpha) c + \text{constant}. \end{aligned}$$

The integral I^* is, therefore, independent of the path in the whole neighborhood $\mathcal{F}$ (IV, 106).

15. Relation of Kneser field theory for the problem of Mayer in parametric form to the field theory developed in this paper.

Let E be a non-singular extremal which is the member for values $t_1 \leq t \leq t_2$, $a_{10}, \dots, a_{n-2,0}$ of an $(n-2)$ -parameter family of extremals

$$(15:1) \quad y_1 = y_1(t, a_1, \dots, a_{n-2}), \quad z_1 = z_1(t, a_1, \dots, a_{n-2})$$

which is such that along each extremal the matrix

$$(15:2) \quad \left\| \begin{array}{cc} y_{1t} & y_{1a_j} \end{array} \right\| \quad (i=1, \dots, n; j=1, \dots, n-2)$$

has rank $n-1$ and the equations

$$(15:3) \quad F_{y_1'} y_{1a_j} = 0$$

are identically satisfied. Then the family of extremals (15:1) forms a field in the sense of Kneser (IX, 43)

As in the case of the non-parametric problem, the function F is identically zero, and it follows readily with the aid of the Euler-Lagrange equations (3:9) that $F_{y_1^i} y_{1a_j} = \text{constant}$ along each extremal (III, 307). Accordingly, the expressions on the left of equations (15:3) vanish identically if they vanish at one point.

THEOREM 15:1. Along each extremal of a Kneser field, the equation

$$F_{y_1^i} dy_1 = 0$$

holds identically where $dy_1 = y_1^i dt + y_{1a_k} da_k$. Furthermore, the extremals of a Kneser field lie on a surface which satisfies the Hamilton-Jacobi partial differential equation $H(y, W_y) = 0$.

The first statement is an immediate consequence of the definition of a Kneser field and the fact that the function F is identically zero along each extremal. The analogue of this condition for a Mayer field in the sense of Bliss is the vanishing of the integral I^* along each extremal of a field. Kneser showed that the surface obtained by eliminating (t, a) from the first n equations in (15:1) satisfies the Hamilton-Jacobi equation. This last result also follows readily from Theorems 13:1, 13:5, and the following:

THEOREM 15:2. A Mayer field in the sense of Bliss consists of a 1-parameter family of Kneser fields.

This theorem is an immediate consequence of Theorem 13:5. The condition (13:4) which must hold on each transversal surface of a Mayer field in the sense of Bliss is the condition (15:3) for a Kneser field.

THEOREM 15:3. Every Kneser field can be imbedded in a one-parameter family of Kneser fields forming a Mayer field in the

sense of Bliss, at least if the Kneser field is sufficiently small.

Let E be a non-singular extremal arc imbedded for values $t_{10} \leq t \leq t_{20}$, α_{r0} in an $(n-2)$ -parameter family of extremals

$$y_1 = y_1(t, \alpha_1, \dots, \alpha_{n-2}), \quad z_1 = z_1(t, \alpha_1, \dots, \alpha_{n-2})$$

which form a Kneser field $\mathcal{H}'$ for (α) in a region $\mathcal{A}$. Set

$$(15:4) \quad A_1(\alpha) = y_1(t_0, \alpha), \quad B_1(\alpha) = z_1(t_0, \alpha)$$

and denote their values at $t = t_0$, $\alpha_r = \alpha_{r0}$ by a_{10} , b_{10} , respectively. From the definition of Kneser fields it follows that

$$(15:5) \quad B_1 A_1 \alpha_r = 0 \quad (r = 1, \dots, n-2),$$

and the matrix $\| y_1' \quad A_1 \alpha_r \|$ has rank $n-1$. As was seen in Section 4 the extremal E is imbedded for values $t_{10} \leq t \leq t_{20}$, a_{10} , b_{10} in a $2n$ -parameter family of extremals

$$(15:6) \quad y_1 = y_1(t, a, b), \quad z_1 = z_1(t, a, b)$$

of the type described in Theorem 4:1 for problem C, where a_1 , b_1 are the values of y_1 , z_1 at $t = t_0$, the parameters b_1 do not vanish simultaneously, and $H(a_{10}, b_{10}) = 0$.

Let $2n$ functions $a_1(\alpha, c)$, $b_1(\alpha, c)$ be defined by the equations

$$(15:7) \quad a_1(\alpha, c) = A_1(\alpha) + c B_1(\alpha)$$

$$(15:8) \quad b_1(A_1 \alpha_r + c B_1 \alpha_r) = 0, \quad b_1 B_1 = B_1(\alpha) B_1(\alpha), \quad H[a(\alpha, c), b] = 0$$

$$(r = 1, \dots, n-2)$$

where $A_1(\alpha)$, $B_1(\alpha)$ are the functions in (15:4). With the aid of conditions (15:5), it is seen that the equations (15:7) and (15:8) are satisfied at the special values

$$[\alpha_r, c, a_1, b_1] = [\alpha_{r0}, 0, a_{10}, b_{10} = B_1(\alpha_{r0})].$$

The functional determinant of the left members of equations (15:8) with respect to the variables b_1 is the determinant

$$\begin{vmatrix} A_1\alpha_r + cB_1\alpha_r & B_1 & y_1' \end{vmatrix}$$

with $H_{z_1} = y_1'$. For the special values given above, it can be shown that this determinant is different from zero by a method analogous to that used in the proof of Theorem 14:3 with the aid of the homogeneity condition $z_1 H_{z_1} = H$. Therefore, the equations (15:8) have solutions $b_1(\alpha, c)$ for (α, c) in a neighborhood of the values $(\alpha_{r0}, 0)$ which are such that $b_1(\alpha_{r0}, 0) = b_{10}$ and $b_1(\alpha, 0) = B_1(\alpha)$.

The $(n-1)$ -parameter family of extremals

$$\begin{aligned} y_1 &= y_1[t, A(\alpha) + cB(\alpha), b(\alpha, c)] = Y_1(t, \alpha, c), \\ (15:9) \quad z_1 &= z_1[t, A(\alpha) + cB(\alpha), b(\alpha, c)] = Z_1(t, \alpha, c), \end{aligned}$$

obtained from the family (15:6) by substituting $a_1(\alpha, c)$, $b_1(\alpha, c)$ for a_1 , b_1 , respectively, contains E for $\alpha_r = \alpha_{r0}$, $c = 0$, and will be shown to have the properties required for the theorem. By a method analogous to that used in the proof of Theorem 14:3 with the aid of the homogeneity condition $z_1 H_{z_1} = H$, the conditions (15:5), (15:7) and (15:8), it is possible to show that the determinant $|Y_{1t} \ Y_1\alpha_r \ Y_{1c}|$ is different from zero along E . Hence, the first n equations in (15:9) have solutions

$$(15:10) \quad t = t(y), \quad \alpha_r = \alpha_r(y), \quad c = c(y) \quad (r=1, \dots, n-2)$$

which have continuous partial derivatives of the first order for (y) in a neighborhood of the values (y) on E .

The family (15:9) defines a Mayer field in a neighborhood $\mathcal{H}$ of the values (y) on E , with functions $q_1(y)$ defined by the equations

$$q_1(y) = Z_1[t(y), \alpha_r(y), c(y)],$$

the right members of which are obtained by substituting the functions (15:10) in the right members of the second group of equations in (15:9). To establish this statement, it is seen with the aid of the equations (15:7) and (15:8) that on the surface $y_1 = A_1(\alpha) + cB_1(\alpha)$ the integral I^* assumes the form

$$\begin{aligned} \int_G q_1(y) dy_1 &= \int_G b_1(\alpha, c) [A_1 \alpha_r + cB_1 \alpha_r] d\alpha_r + b_1(\alpha, c) B_1 dc \\ &= B_1(\alpha) B_1(\alpha) c + \text{constant}. \end{aligned}$$

It follows that the integral I^* is independent of the path in the whole neighborhood $\mathcal{H}$ (V, 215).

Moreover, for fixed values of c and (α) in a neighborhood of the values $(\alpha_{r0}, 0)$ belonging to E , the family (15:9) defines a Kneser field. This statement can be proved with the aid of the equations (15:7) and (15:8) by a method analogous to that used in the proof of Theorem 14:3.

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VITA

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